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On the Thermal Instability of a Rotating-Fluid Sphere Containing Heat Sources

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ON THE THERMAL INSTABILITY OF A ROTATING-FLUID SPHERE CONTAINING HEAT SOURCES

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CONTENTS

The theory of marginal convection in a uniformly rotating, self-gravitating, fluid sphere, of uniform density and containing a uniform distribution of heat sources, is developed to embrace modes of convection which are asymmetric with respect to the axis of rotation. It is shown that these modes are the most unstable, except for the smallest Taylor numbers, $\mathbf{T}$ (a measure of the rotation rate); i.e. for any $\mathbf{T}$ and ω (Prandtl number), the lowest Rayleigh number (a measure of the temperature gradients in the sphere) is associated with an asymmetric motion. This is demonstrated both by an expansion method suitable for small $\mathbf{T}$, and by asymptotic theory for $\mathbf{T}\rightarrow\infty$. For large $\mathbf{T}$, the eigenmode most easily excited is small in amplitude everywhere except near a cylindrical surface, of radius about half that of the sphere, and coaxial with the diameter parallel to the angular velocity vector.

1. Introduction

The dynamo hypothesis provides one of the most attractive explanations of the magnetic field of cosmic bodies, such as the Earth. It is known theoretically that such homogeneous dynamos cannot maintain axially symmetric fields, and it is perhaps significant that one of the best studied cosmic fields, that of the Earth, while showing a high degree of symmetry about its polar axis, exhibits significant asymmetries. The dynamo theory is, therefore, probably† untenable unless we can find a plausible reason for asymmetric fluid motions within the body. Now one of the most obvious forces driving the fluid is the buoyancy created by internal heating; in the case of the Earth, for example, by dissolved radioactivity in the core. There seems, however, no reason to suppose that these sources are distributed in any but a spherically symmetric fashion, giving a buoyancy force which is radial and spherically symmetric everywhere. At first sight, it seems most unlikely that a radial buoyancy force, together with the axially symmetric effects of rotation, could together result in asymmetric motions. On reflection, however, we may recall a celebrated case in which rotation induces asymmetry, namely the formation of the Jacobi ellipsoid from the

† I say ‘probably’ since it has not been proved that asymmetric fields could not be maintained by axisymmetric motions.

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Maclaurin spheroid beyond the point of bifurcation of the sequence of rapidly rotating, self-gravitating, uniform fluid bodies. Also we may recall the observational fact that, under certain circumstances, asymmetric motions are observed in annulus experiments (in which free convection between two coaxial, differentially heated, cylinders rotating together is studied).

One can give general reasons why asymmetric motions may be expected in a rotating fluid sphere containing heat sources. To simplify the discussion, we henceforward ignore magnetic effects on the motion, and consider the marginally stable state only, i.e. seek, for given Taylor number ($\mathbf{T}$) and Prandtl number (ω), the smallest Rayleigh number (R) at which persistent motions of infinitesimal amplitude are possible. Here

$$\mathbf{T} = \left(\frac{2\Omega r_0^2}{\nu}\right)^2, \quad \omega = \frac{\nu}{\kappa}, \quad R = \frac{g\alpha\beta}{\nu\kappa} r_0^4, \tag{1·1}$$

where r_0 is the radius of the sphere, $\mathbf{\Omega}$ is its angular velocity, ν is the kinematic viscosity, κ is the thermal diffusivity, α is the coefficient of volume expansion, $-g\mathbf{r}/r_0$ is the acceleration due to gravity at a point distant $\mathbf{r}$ from its centre, and $-\beta\mathbf{r}/r_0$ is the temperature gradient at that point. To gain qualitative insight into convection in the rotating case, it seems profitable to draw analogies with the (better-studied) case of Bénard convection in a rotating fluid layer heated from below and cooled from above (cf. Chandrasekhar 1961, chap. III). The argument is most clear-cut if we suppose that ω is fixed, $\mathbf{T}$ is large, and R is increased slowly and systematically from zero.

We consider first the possibility that the heat from the interior of the sphere escapes to the surface by motions which convect it parallel to the axis of rotation towards the poles. The effective driving force of such motions is the component of buoyancy parallel to the axis, and the situation becomes akin to a Bénard layer in which $\mathbf{\Omega}$ is perpendicular to the layer. It has been shown by Chandrasekhar (1961) that convection, when it first occurs in such a layer, will be in the form of an array of cells, each of thickness $\mathbf{T}^{-\frac{1}{6}}$, and that the Rayleigh number will then be $O(\mathbf{T}^{\frac{2}{3}})$. In taking this fact over to the spherical case, one has to make allowances for curvature effects. A simple argument (cf. Roberts 1965*a*, p. 245; this paper is henceforward called 'paper I') shows that convection of this type, when it first occurs in the sphere, will be confined to a single axial cell, of thickness $\mathbf{T}^{-\frac{1}{6}}$, and that the Rayleigh number will also be $O(\mathbf{T}^{\frac{2}{3}})$.

On the other hand, we might consider the possibility that the heat from the interior of the sphere escapes to the surface by motions which convect it perpendicular to the axis of rotation towards the equator. The effective driving force of such motions is the component of buoyancy perpendicular to the axis, and the situation becomes akin to a Bénard layer in which $\mathbf{\Omega}$ is parallel to the layer. It is easily shown that convection, when it first occurs in such a layer, takes the form of infinitely long rolls with axes parallel to $\mathbf{\Omega}$, and that the Rayleigh number at which this occurs is exactly the same as for the non-rotating layer. Taking this fact over to the spherical cases, we might surmise that modes of convection proportional to $\cos m\phi$ and $\sin m\phi$, where ϕ is the aximuthal angle, may exist in which m is large, in which the variations in z are comparatively slow or absent, and for which R is virtually unaltered from its value, at the same m, in the non-rotating case. It is true that, in the non-rotating situation, $R \sim m^4$ for large m, so that the motions will certainly have a higher critical Rayleigh number than the axial cell if $m > O(\mathbf{T}^{\frac{1}{6}})$, but one might feel that, provided this inequality is violated, these motions will provide the smallest Rayleigh number in the rotating system. Again, however, one must make allowances for curvature effects in the spherical case. These prevent the formation of the infinitely long rolls crucial to the result for the plane layer, and once they are 'cut off' to 'fit into' the sphere, the critical value of R is increased drastically, particularly for small m.

It will be shown in this paper that the most unstable modes in a sphere are intermediate to the two extreme cases considered above, and that the motions are a compromise between the axial cell appropriate for small m, and the equatorial cell appropriate for larger m. For large $\mathbf{T}$, the critical value of m is $O(\mathbf{T}^{\frac{1}{6}})$; the critical R is $O(\mathbf{T}^{\frac{2}{3}})$, but is smaller than that of the axial cell of paper I. These findings are corroborated by a study of the case of small $\mathbf{T}$, from which it is shown that the value of m, for which R is least, is zero only if $\mathbf{T}$ is small. As $\mathbf{T}$ increases, the critical m also increases systematically. One point must, however, be made clear: these facts are demonstrated in particular cases of ω only, and it is not inconceivable that they become invalid in the $\omega \to 0$ or $\omega \to \infty$ extremes. Also, of course, non-linear effects which might conceivably give rise to subcritical finite-amplitude motions (Veronis 1966) are ignored.

Before leaving this qualitative discussion I return briefly to the dynamo problem. In his study of nearly symmetric dynamos at large magnetic Reynolds numbers, Braginskiĭ (1964) has shown that, not only must the velocity fields, $\mathbf{u}$, be asymmetric, they must be so asymmetric that the average over ϕ of $\mathbf{u} \times \partial\mathbf{u}/\partial\phi$ is non-zero. (For more precise details, see Braginskiĭ's paper.) The motions studied in this paper satisfy this demand.

The convection problem discussed in this paper has also been considered by Chandrasekhar (1957, 1961), Bisshopp (1958), Roberts (paper I) and Bisshopp & Niiler (1965). Related work has been reported by Chamalaun & Roberts (1962) and Roberts (1965*b*).

2. Derivation of scalar equations

The vector form of the perturbation equations governing marginal convection may be obtained by reference to chapter 9 of Chandrasekhar's treatise (1961). In non-dimensional form, the radius of the sphere providing the unit of length, they are

$$\frac{\partial \mathbf{u}}{\partial t} + \lambda \mathbf{1}_z \times \mathbf{u} = -\operatorname{grad} \varpi + R\Theta\mathbf{x} + \nabla^2\mathbf{u}, \tag{2·1}$$

$$\omega \frac{\partial \Theta}{\partial t} = \mathbf{u} . \mathbf{x} + \nabla^2\Theta, \tag{2·2}$$

$$\operatorname{div} \mathbf{u} = 0. \tag{2·3}$$

Here R is the Rayleigh number, $\mathbf{T} = \lambda^2$ is the Taylor number, and ω is the Prandtl number. The variables $\mathbf{u}$, Θ, and ϖ denote, respectively, the fluid velocity, the temperature perturbation, and the pressure perturbation, all measured in suitably scaled units. The unit vector, $\mathbf{1}_z$, is parallel to the axis of rotation, Oz; $\mathbf{x}$ is the position vector drawn from the centre, O, of the sphere as origin. For a fluid contained by a rigid shell, the boundary conditions which solutions to (2·1) to (2·3) must obey are

$$\Theta = 0, \quad \mathbf{u} = 0, \quad \text{on} \quad r = 1, \tag{2·4r}$$

and similar conditions must be applied if the surface of the fluid is free (see (2·13f) below).

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We may derive three independent scalar equations from (2·1) by subjecting it in turn to the operators div, $\mathbf{x}.\text{curl}$ and $\mathbf{x}.\text{curl}^2$. The first of these equations involves ϖ, while the others do not; i.e. it essentially determines ϖ when the other field quantities are known. Since, however, we have no boundary condition to apply to ϖ, we can, as far as solving the stability problem is concerned, omit this equation from our discussion. We therefore pass on to the remaining two equations: these are

$$\mathbf{x}.\text{curl}\left(\frac{\partial}{\partial t}-\nabla^2\right)\mathbf{u}=\lambda\mathbf{x}.\frac{\partial\mathbf{u}}{\partial z}, \tag{2·5}$$

$$\mathbf{x}.\nabla^2\left(\frac{\partial}{\partial t}-\nabla^2\right)\mathbf{u}=-\lambda\mathbf{x}.\frac{\partial}{\partial z}\text{curl}\,\mathbf{u}-RL^2\Theta, \tag{2·6}$$

where L^2 is the operator

$$L^2=\sum_i\left[x_i\frac{\partial}{\partial x_i}+\left(x_i\frac{\partial}{\partial x_i}\right)^2\right]-x^2\nabla^2. \tag{2·7}$$

If we now write, as suggested by (2·3), $\mathbf{u}$ as the sum of its toroidal and poloidal parts,

$$\mathbf{u}=\text{curl}\,T\mathbf{x}+\text{curl}^2 S\mathbf{x}, \tag{2·8}$$

equations (2·5), (2·6) and (2·2) reduce to

$$\left[L^2\left(\frac{\partial}{\partial t}-\nabla^2\right)-\lambda\frac{\partial}{\partial\phi}\right]T=-\lambda Q^3S, \tag{2·9}$$

$$\left[L^2\left(\frac{\partial}{\partial t}-\nabla^2\right)-\lambda\frac{\partial}{\partial\phi}\right]\nabla^2S=\lambda Q^3T-RL^2\Theta, \tag{2·10}$$

$$\left(\omega\frac{\partial}{\partial t}-\nabla^2\right)\Theta=L^2S, \tag{2·11}$$

where Q^3 is the operator

$$Q^3=\frac{\partial}{\partial z}-\frac{1}{2}\left(L^2\frac{\partial}{\partial z}+\frac{\partial}{\partial z}L^2\right), \tag{2·12}$$

which commutes with ∇^2, but not with L^2. The variable ϕ is the azimuthal component of spherical polar coordinates (r,θ,ϕ) in which $\theta=0$ is the axis of rotation. The boundary conditions may now be written

$$S=\frac{\partial S}{\partial r}=T=\Theta=0,\quad\text{on}\quad r=1, \tag{2·13r}$$

if the bounding surface is rigid, and

$$S=\frac{\partial^2S}{\partial r^2}=\frac{\partial T}{\partial r}-T=\Theta=0,\quad\text{on}\quad r=1, \tag{2·13f}$$

if it is free (and if the time-scale of the motion is large compared with the period of surface gravity waves, the usual case considered in convection problems). The four boundary conditions (2·13), together with the requirement that the solution shall be bounded within the sphere (and in particular, at $r=0$), suffice to determine the characteristic values. (The system (2·9) to (2·11) is eighth order in $\partial/\partial r$ despite appearances: since L^2 involves $\partial/\partial\theta$ and $\partial/\partial\phi$, but not $\partial/\partial r$, Q^3 is first order in $\partial/\partial r$, and not third order.)

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3. The basis for the expansion of S

We proceed as follows. We expand S in an infinite series, each term of which satisfies the first two conditions (2·13). Then equation (2·11) is solved subject to the last of conditions (2·13); likewise, equation (2·9) is solved under the third condition of (2·13). The resulting expressions for Θ and T are used to evaluate the right-hand side of (2·10). On multiplying (2·10) successively by each function used in the expansion for S, and integrating over the interior of the sphere, an infinite set of linear homogeneous equations are obtained for the coefficients of the S-expansion. This set possesses non-trivial solutions, if, and only if, R takes one of an infinite discrete set of characteristic values, of which we only aim to determine the smallest. (See also Chandrasekhar & Reid 1957.)

We expand S in the form

$$S = \sum_{n=m}^{\infty} \sum_{i=1}^{\infty} A_{ni} B_{ni}(r)\, P_n^m(\mu) \exp\{\mathrm{i}(m\phi + pt)\}, \tag{3·1}$$

where $\mu = \cos\theta$,

$$P_n^m(\mu) = (1-\mu^2)^{\frac{1}{2}m} \frac{\mathrm{d}^m}{\mathrm{d}\mu^m} P_n(\mu) \tag{3·2}$$

is (Ferrer's) associated Legendre function, and

$$B_{ni}(r) = \frac{1}{\sqrt{r}} \left[\frac{J_{n+\frac{1}{2}}(\alpha_{ni} r)}{J_{n+\frac{1}{2}}(\alpha_{ni})} - \frac{I_{n+\frac{1}{2}}(\alpha_{ni} r)}{I_{n+\frac{1}{2}}(\alpha_{ni})} \right]. \tag{3·3}$$

Here J_ν and I_ν denote Bessel functions of the first kind of real and imaginary arguments, and α_{ni} denotes the ith smallest positive root of the equation

$$\frac{J'_{n+\frac{1}{2}}(\alpha_{ni})}{J_{n+\frac{1}{2}}(\alpha_{ni})} - \frac{I'_{n+\frac{1}{2}}(\alpha_{ni})}{I_{n+\frac{1}{2}}(\alpha_{ni})} = -\xi \alpha_{ni}, \tag{3·4}$$

where

$$\xi = \begin{cases} 0, & \text{if } \; r = 1 \text{ is a rigid surface,} \\ 1, & \text{if } \; r = 1 \text{ is a free surface.} \end{cases}$$

The roots of equation (3·4), and the corresponding value of

$$\epsilon_{ni} = \frac{J'_{n+\frac{1}{2}}(\alpha_{ni})}{J_{n+\frac{1}{2}}(\alpha_{ni})} + \frac{I'_{n+\frac{1}{2}}(\alpha_{ni})}{I_{n+\frac{1}{2}}(\alpha_{ni})}, \tag{3·5}$$

have been determined for $n = 1(1)20$ and $i = 1(1)20$ and for $\xi = 0$ and 1; and tables listing these values are available on request.

It is evident, from equations (3·3) and (3·4), that

$$B_{ni}(1) = B'_{ni}(1) = 0, \quad \text{for} \quad \xi = 0, \tag{3·6r}$$
$$B_{ni}(1) = B''_{ni}(1) = 0, \quad \text{for} \quad \xi = 1, \tag{3·6f}$$

so that the boundary conditions (2·13) on S are automatically obeyed.

Associated with the function B_{ni}, there is a related function C_{ni} defined by

$$C_{ni}(r) = \frac{1}{\sqrt{r}} \left[\frac{J_{n+\frac{1}{2}}(\alpha_{ni} r)}{J_{n+\frac{1}{2}}(\alpha_{ni})} + \frac{I_{n+\frac{1}{2}}(\alpha_{ni} r)}{I_{n+\frac{1}{2}}(\alpha_{ni})} \right]. \tag{3·7}$$

We find it convenient to introduce the following abbreviations:

$$\beta_{ni} = \frac{J_{n+\frac{3}{2}}(\alpha_{ni})}{J_{n+\frac{1}{2}}(\alpha_{ni})} - \frac{I_{n+\frac{3}{2}}(\alpha_{ni})}{I_{n+\frac{1}{2}}(\alpha_{ni})}, \tag{3·8}$$

$$\gamma_{ni} = \frac{J_{n-\frac{1}{2}}(\alpha_{ni})}{J_{n+\frac{1}{2}}(\alpha_{ni})} + \frac{I_{n-\frac{1}{2}}(\alpha_{ni})}{I_{n+\frac{1}{2}}(\alpha_{ni})}. \tag{3·9}$$

Note that $$\beta_{ni}+\gamma_{ni} = \frac{2(2n+1)}{\alpha_{ni}}, \quad \beta_{ni}-\gamma_{ni} = -2\epsilon_{ni}.$$

The following properties of B_{ni} and C_{ni} can be established:

$$B_{ni}(1) = 0, \quad B'_{ni}(1) = -\xi\alpha_{ni}^2, \tag{3·10}$$

$$C_{ni}(1) = 2, \quad C'_{ni}(1) = 2n-\alpha_{ni}\beta_{ni} = \alpha_{ni}\gamma_{ni}-2(n+1), \tag{3·11}$$

$$\left.\begin{aligned}\nabla^2[B_{ni}P_n^m\,\mathrm{e}^{\mathrm{i}m\phi}] &= -\alpha_{ni}^2 C_{ni}P_n^m\,\mathrm{e}^{\mathrm{i}m\phi},\\ \nabla^2[C_{ni}P_n^m\,\mathrm{e}^{\mathrm{i}m\phi}] &= -\alpha_{ni}^2 B_{ni}P_n^m\,\mathrm{e}^{\mathrm{i}m\phi},\end{aligned}\right\} \tag{3·12}$$

$$(\nabla^4-\alpha_{ni}^4)\,B_{ni}P_n^m\,\mathrm{e}^{\mathrm{i}m\phi} = (\nabla^4-\alpha_{ni}^4)\,C_{ni}P_n^m\,\mathrm{e}^{\mathrm{i}m\phi} = 0, \tag{3·13}$$

$$\int_0^1 B_{ni}B_{nj}r^2\,\mathrm{d}r = \tfrac{1}{4}\delta_{ij}(4-4\xi+\xi\alpha_{ni}\beta_{ni}-\xi\alpha_{ni}\gamma_{ni}), \tag{3·14}$$

$$\alpha_{nj}^2\int_0^1 B_{ni}C_{nj}r^2\,\mathrm{d}r = \alpha_{ni}^2\int_0^1 B_{nj}C_{ni}r^2\,\mathrm{d}r = \tfrac{1}{4}\delta_{ij}\,\alpha_{ni}^2(\xi\alpha_{ni}^2-\beta_{ni}\gamma_{ni}) + \frac{2\alpha_{ni}^2\alpha_{nj}^2}{\alpha_{ni}^4-\alpha_{nj}^4}(\alpha_{ni}\beta_{ni}-\alpha_{nj}\beta_{nj})\,(1-\delta_{ij}), \tag{3·15}$$

$$\begin{aligned}\int_0^1\frac{1}{\sqrt{r}}\frac{J_{n+\frac{1}{2}}(xr)}{J_{n+\frac{1}{2}}(x)}B_{ni}r^2\,\mathrm{d}r &= \frac{\alpha_{ni}^2}{\alpha_{ni}^4-x^4}\left[\alpha_{ni}\beta_{ni}+\xi x^2-\frac{2xJ_{n+\frac{3}{2}}(x)}{J_{n+\frac{1}{2}}(x)}\right]\\ &= -\frac{\alpha_{ni}^2}{\alpha_{ni}^4-x^4}\left[\alpha_{ni}\gamma_{ni}-\xi x^2-\frac{2xJ_{n-\frac{1}{2}}(x)}{J_{n+\frac{1}{2}}(x)}\right],\end{aligned} \tag{3·16}$$

$$\int_0^1\frac{1}{\sqrt{r}}\left[\frac{J_{n+\frac{1}{2}}(\alpha_{n-2,j}r)}{J_{n-\frac{3}{2}}(\alpha_{n-2,j})}+\frac{I_{n+\frac{1}{2}}(\alpha_{n-2,j}r)}{I_{n-\frac{3}{2}}(\alpha_{n-2,j})}\right]B_{ni}(r)r^2\,\mathrm{d}r = \frac{(2n-1)\,\alpha_{ni}^2}{\alpha_{n-2,j}(\alpha_{ni}^4-\alpha_{n-2,j}^4)}\,[\xi\alpha_{n-2,j}^3-\alpha_{ni}\gamma_{ni}\beta_{n-2,j}], \tag{3·17}$$

$$\begin{aligned}&\int_0^1\frac{1}{\sqrt{r}}\left[\frac{J_{n+\frac{1}{2}}(\alpha_{n-2,j}r)}{J_{n-\frac{3}{2}}(\alpha_{n-2,j})}-\frac{I_{n+\frac{1}{2}}(\alpha_{n-2,j}r)}{I_{n-\frac{3}{2}}(\alpha_{n-2,j})}\right]B_{ni}(r)\,r^2\,\mathrm{d}r\\ &\qquad= \frac{\alpha_{ni}^2}{\alpha_{ni}^4-\alpha_{n-2,j}^4}\,[(2-2n\xi+\xi)\,\alpha_{ni}\gamma_{ni}+(2+2n\xi-\xi)\,\alpha_{n-2,j}\beta_{n-2,j}],\end{aligned} \tag{3·18}$$

$$\int_0^1\frac{1}{\sqrt{r}}\left[\frac{J_{n+\frac{1}{2}}(\alpha_{n+2,j}r)}{J_{n+\frac{5}{2}}(\alpha_{n+2,j})}+\frac{I_{n+\frac{1}{2}}(\alpha_{n+2,j}r)}{I_{n+\frac{5}{2}}(\alpha_{n+2,j})}\right]B_{ni}(r)\,r^2\,\mathrm{d}r = \frac{(2n+3)\,\alpha_{ni}^2}{\alpha_{n+2,j}(\alpha_{n+2,j}^4-\alpha_{ni}^4)}\,[\xi\alpha_{n+2,j}^3-\alpha_{ni}\beta_{ni}\gamma_{n+2,j}], \tag{3·19}$$

$$\begin{aligned}&\int_0^1\frac{1}{\sqrt{r}}\left[\frac{J_{n+\frac{1}{2}}(\alpha_{n+2,j}r)}{J_{n+\frac{5}{2}}(\alpha_{n+2,j})}-\frac{I_{n+\frac{1}{2}}(\alpha_{n+2,j}r)}{I_{n+\frac{5}{2}}(\alpha_{n+2,j})}\right]B_{ni}(r)\,r^2\,\mathrm{d}r\\ &\qquad= \frac{\alpha_{ni}^2}{\alpha_{n+2,j}^4-\alpha_{ni}^4}\,[(2-2n\xi-3\xi)\,\alpha_{n+2,j}\gamma_{n+2,j}+(2+2n\xi+3\xi)\,\alpha_{ni}\beta_{ni}].\end{aligned} \tag{3·20}$$

By (3·14) and the usual expression for the integral over $\mathrm{d}\mu\,\mathrm{d}\phi$ of the product of two spherical harmonics, we see that the functions $B_{sj}P_s^q\,\mathrm{e}^{\mathrm{i}(q\phi+pt)}$ and $B_{ni}P_n^m\,\mathrm{e}^{-\mathrm{i}(m\phi+pt)}$ are orthogonal: the integral over the interior of the unit sphere of their product vanishes unless $q=m$, $s=n$ and $j=i$ simultaneously.

Writing L^2 in the form

$$L^2 = -\left[\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial}{\partial\mu}\right\} + \frac{1}{1-\mu^2}\frac{\partial^2}{\partial\phi^2}\right], \tag{3·21}$$

and recalling that P_n^m obeys the differential equation

$$\left[\frac{\mathrm{d}}{\mathrm{d}\mu}\left\{(1-\mu^2)\frac{\mathrm{d}}{\mathrm{d}\mu}\right\} - \frac{m^2}{1-\mu^2}\right] P_n^m(\mu) = -n(n+1)\,P_n^m(\mu), \tag{3·22}$$

we see that, for an arbitrary function, f, of r,

$$L^2[f(r)\,P_n^m(\mu)\,\mathrm{e}^{\mathrm{i}m\phi}] = n(n+1)\,[f(r)\,P_n^m(\mu)\,\mathrm{e}^{\mathrm{i}m\phi}]. \tag{3·23}$$

Writing Q^3 in the form

$$Q^3 = \left[\frac{\partial}{\partial\mu}\left\{\mu(1-\mu^2)\frac{\partial}{\partial\mu}\right\} + \frac{\mu}{1-\mu^2}\frac{\partial^2}{\partial\phi^2}\right]\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) - L^2\left[(1-\mu^2)\frac{\partial}{\partial\mu}\right], \tag{3·24}$$

and recalling that P_n^m obeys the recurrence relations

$$\left[\frac{\mathrm{d}}{\mathrm{d}\mu}\left\{\mu(1-\mu^2)\frac{\mathrm{d}}{\mathrm{d}\mu}\right\} - \frac{m^2\mu}{1-\mu^2}\right] P_n^m(\mu)$$

$$= -\frac{n(n+2)\,(n-m+1)}{2n+1}\,P_{n+1}^m(\mu) - \frac{(n-1)\,(n+1)\,(n+m)}{2n+1}\,P_{n-1}^m(\mu), \tag{3·25}$$

$$(1-\mu^2)\frac{\mathrm{d}P_n^m(\mu)}{\mathrm{d}\mu} = -\frac{n(n-m+1)}{2n+1}\,P_{n+1}^m(\mu) + \frac{(n+1)\,(n+m)}{2n+1}\,P_{n-1}^m(\mu), \tag{3·26}$$

we see that

$$Q^3\left[\frac{1}{\sqrt{r}}J_{n+\frac12}(xr)\,P_n^m\mathrm{e}^{\mathrm{i}m\phi}\right] = \frac{xn(n+2)\,(n-m+1)}{2n+1}\frac{1}{\sqrt{r}}J_{n+\frac32}(xr)\,P_{n+1}^m\mathrm{e}^{\mathrm{i}m\phi}$$

$$-\frac{x(n-1)\,(n+1)\,(n+m)}{2n+1}\frac{1}{\sqrt{r}}J_{n-\frac12}(xr)\,P_{n-1}^m\mathrm{e}^{\mathrm{i}m\phi}, \tag{3·27}$$

so that, in particular,

$$Q^3[B_{ni}P_n^m\mathrm{e}^{\mathrm{i}m\phi}] = \frac{n(n+2)\,(n-m+1)}{2n+1}\alpha_{ni}\frac{1}{\sqrt{r}}\left[\frac{J_{n+\frac32}(\alpha_{ni}r)}{J_{n+\frac12}(\alpha_{ni})} + \frac{I_{n+\frac32}(\alpha_{ni}r)}{I_{n+\frac12}(\alpha_{ni})}\right]P_{n+1}^m\mathrm{e}^{\mathrm{i}m\phi}$$

$$-\frac{(n-1)\,(n+1)\,(n+m)}{2n+1}\alpha_{ni}\frac{1}{\sqrt{r}}\left[\frac{J_{n-\frac12}(\alpha_{ni}r)}{J_{n+\frac12}(\alpha_{ni})} - \frac{I_{n-\frac12}(\alpha_{ni}r)}{I_{n+\frac12}(\alpha_{ni})}\right]P_{n-1}^m\mathrm{e}^{\mathrm{i}m\phi}, \tag{3·28}$$

$$Q^3\left[\frac{1}{\sqrt{r}}\left\{\frac{J_{n+\frac32}(\alpha_{ni}r)}{J_{n+\frac12}(\alpha_{ni})} \mp \frac{I_{n+\frac32}(\alpha_{ni}r)}{I_{n+\frac12}(\alpha_{ni})}\right\}P_{n+1}^m\,\mathrm{e}^{\mathrm{i}m\phi}\right] = \frac{\alpha_{ni}(n+1)\,(n+3)\,(n-m+2)}{2n+3}$$

$$\times\left[\frac{1}{\sqrt{r}}\left\{\frac{J_{n+\frac52}(\alpha_{ni}r)}{J_{n+\frac12}(\alpha_{ni})} \pm \frac{I_{n+\frac52}(\alpha_{ni}r)}{I_{n+\frac12}(\alpha_{ni})}\right\}P_{n+2}^m\mathrm{e}^{\mathrm{i}m\phi}\right] - \frac{\alpha_{ni}n(n+2)\,(n+m+1)}{2n+3}\begin{Bmatrix}B_{ni}\\C_{ni}\end{Bmatrix}P_n^m\mathrm{e}^{\mathrm{i}m\phi}, \tag{3·29}$$

$$Q^3\left[\frac{1}{\sqrt{r}}\left\{\frac{J_{n-\frac12}(\alpha_{ni}r)}{J_{n+\frac12}(\alpha_{ni})} \pm \frac{I_{n-\frac12}(\alpha_{ni}r)}{I_{n+\frac12}(\alpha_{ni})}\right\}P_{n-1}^m\,\mathrm{e}^{\mathrm{i}m\phi}\right] = \frac{\alpha_{ni}\,(n-1)\,(n+1)\,(n-m)}{2n-1}\begin{Bmatrix}B_{ni}\\C_{ni}\end{Bmatrix}P_n^m\mathrm{e}^{\mathrm{i}m\phi}$$

$$-\frac{\alpha_{ni}n(n-2)\,(n+m-1)}{2n-1}\left[\frac{1}{\sqrt{r}}\left\{\frac{J_{n-\frac32}(\alpha_{ni}r)}{J_{n+\frac12}(\alpha_{ni})} \pm \frac{I_{n-\frac32}(\alpha_{ni}r)}{I_{n+\frac12}(\alpha_{ni})}\right\}P_{n-2}^m\mathrm{e}^{\mathrm{i}m\phi}\right]. \tag{3·30}$$

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(Note: relations (3·25) to (3·30) are valid in the case $n = m$ provided we adopt the convention that $P_n^m \equiv 0$ for $n < m$.) In the following sections, the following abbreviations are useful:

$$a_{ni} = \tfrac{1}{4}(4-4\xi+\xi\alpha_{ni}\beta_{ni}-\xi\alpha_{ni}\gamma_{ni}), \tag{3·31}$$

$$c_{nij} = \begin{cases} \dfrac{1}{4}\left(\xi - \dfrac{\beta_{ni}\gamma_{ni}}{\alpha_{ni}^2}\right) & (i=j), \quad (3·32) \\ \dfrac{2(\alpha_{ni}\beta_{ni}-\alpha_{nj}\beta_{nj})}{\alpha_{ni}^4-\alpha_{nj}^4} & (i \neq j), \quad (3·33) \end{cases}$$

$$m_n(x) = \frac{xJ_{n+\frac{1}{2}}(x)}{J_{n+\frac{3}{2}}(x)}, \tag{3·34}$$

$$p_n(x) = \frac{xJ_{n+\frac{1}{2}}(x)}{J_{n-\frac{1}{2}}(x)}, \tag{3·35}$$

$$m_{ni}(x) = \frac{1}{\alpha_{ni}^4-x^4}\left[2x^2-(\alpha_{ni}\beta_{ni}+\xi x^2)\frac{xJ_{n+\frac{1}{2}}(x)}{J_{n+\frac{3}{2}}(x)}\right], \tag{3·36}$$

$$p_{ni}(x) = \frac{1}{\alpha_{ni}^4-x^4}\left[2x^2-(\alpha_{ni}\gamma_{ni}-\xi x^2)\frac{xJ_{n+\frac{1}{2}}(x)}{J_{n-\frac{1}{2}}(x)}\right], \tag{3·37}$$

$$\chi^2 = -\mathrm{i}p\omega, \tag{3·38}$$

$$q_n^2 = -\mathrm{i}\left[p-\frac{m\lambda}{n(n+1)}\right], \tag{3·39}$$

$$f_n = \left[\frac{n+2}{2n+1}p_{n+1}(q_n)-\frac{n-1}{2n+1}m_{n-1}(q_n)\right]^{-1}. \tag{3·40}$$

It may be noted that

$$m_n(x)\,p_{n+1}(x) = x^2, \qquad \frac{1}{m_n(x)}+\frac{1}{p_n(x)} = \frac{2n+1}{x^2}, \tag{3·41}$$

$$\frac{m_{ni}}{m_n}+\frac{p_{ni}}{p_n} = 0. \tag{3·42}$$

4. Expressions for Θ and T

We now solve equation (2·11) subject to the last of conditions (2·13). It may be verified, by direct substitution, that a particular integral of equation (2·11) is

$$\Theta^{(p)} = \sum_{n=m}^{\infty}\sum_{i=1}^{\infty}\frac{A_{ni}}{\sqrt{r}}\left[\frac{n(n+1)}{\alpha_{ni}^2-\chi^2}\frac{J_{n+\frac{1}{2}}(\alpha_{ni}r)}{J_{n+\frac{1}{2}}(\alpha_{ni})}+\frac{n(n+1)}{\alpha_{ni}^2+\chi^2}\frac{I_{n+\frac{1}{2}}(\alpha_{ni}r)}{I_{n+\frac{1}{2}}(\alpha_{ni})}\right]P_n^m(\mu)\,\mathrm{e}^{\mathrm{i}(m\phi+pt)}$$

$$= \sum_{n=m}^{\infty}\sum_{i=1}^{\infty}\frac{n(n+1)A_{ni}}{\alpha_{ni}^4-\chi^4}[\chi^2B_{ni}(r)+\alpha_{ni}^2C_{ni}(r)]\,P_n^m(\mu)\,\mathrm{e}^{\mathrm{i}(m\phi+pt)}. \tag{4·1}$$

It is clear that, since

$$\Theta^{(p)}(1) = \sum_{n=m}^{\infty}\sum_{i=1}^{\infty}\frac{2n(n+1)\,\alpha_{ni}^2}{\alpha_{ni}^4-\chi^4}A_{ni}P_n^m(\mu)\,\mathrm{e}^{\mathrm{i}(m\phi+pt)}, \tag{4·2}$$

this solution does not in general satisfy (2·13). We must therefore add to $\Theta^{(p)}$ a multiple,

$\Theta^{(c)}$, of the complementary function defined by equation (2·11), i.e.

$$\frac{1}{\sqrt{r}}\frac{J_{n+\frac{1}{2}}(\chi r)}{J_{n+\frac{1}{2}}(\chi)}P_n^m(\mu)\,\mathrm{e}^{\mathrm{i}(m\phi+pt)}.$$

Evidently, in order that $\Theta(1) \equiv \Theta^{(p)}(1)+\Theta^{(c)}(1) = 0,$

we must take
$$\Theta^{(c)} = -\sum_{n=m}^{\infty}\sum_{i=1}^{\infty}\frac{2n(n+1)\,\alpha_{ni}^2}{\alpha_{ni}^4-\chi^4}A_{ni}\frac{1}{\sqrt{r}}\frac{J_{n+\frac{1}{2}}(\chi r)}{J_{n+\frac{1}{2}}(\chi)}P_n^m(\mu)\,\mathrm{e}^{\mathrm{i}(m\phi+pt)}, \tag{4·3}$$

i.e.
$$\Theta = \sum_{n=m}^{\infty}\sum_{i=1}^{\infty}\frac{n(n+1)\,A_{ni}}{\alpha_{ni}^4-\chi^4}\left[\chi^2B_{ni}(r)+\alpha_{ni}^2C_{ni}(r)-\frac{2\alpha_{ni}^2}{\sqrt{r}}\frac{J_{n+\frac{1}{2}}(\chi r)}{J_{n+\frac{1}{2}}(\chi)}\right]P_n^m(\mu)\,\mathrm{e}^{\mathrm{i}(m\phi+pt)}. \tag{4·4}$$

Next, we solve equation (2·9) subject to the penultimate condition of (2·13). It may be verified, by direct substitution, that a particular integral of equation (2·9) is

$$\begin{aligned}T^{(p)} = \sum_{n=m}^{\infty}\sum_{i=1}^{\infty}\Bigg\{&\frac{\alpha_{ni}n(n+2)\,(n-m+1)}{2n+1}\left[\frac{K_{ni}}{\sqrt{r}}\left\{\frac{J_{n+\frac{3}{2}}(\alpha_{ni}r)}{J_{n+\frac{1}{2}}(\alpha_{ni})}-\frac{I_{n+\frac{3}{2}}(\alpha_{ni}r)}{I_{n+\frac{1}{2}}(\alpha_{ni})}\right\}\right.\\&\left.+\frac{M_{ni}}{\sqrt{r}}\left\{\frac{J_{n+\frac{3}{2}}(\alpha_{ni}r)}{J_{n+\frac{1}{2}}(\alpha_{ni})}+\frac{I_{n+\frac{3}{2}}(\alpha_{ni}r)}{I_{n+\frac{1}{2}}(\alpha_{ni})}\right\}\right]P_{n+1}^m(\mu)\,\mathrm{e}^{\mathrm{i}(m\phi+pt)}\\&-\frac{\alpha_{ni}(n-1)\,(n+1)\,(n+m)}{2n+1}\left[\frac{N_{ni}}{\sqrt{r}}\left\{\frac{J_{n-\frac{1}{2}}(\alpha_{ni}r)}{J_{n+\frac{1}{2}}(\alpha_{ni})}+\frac{I_{n-\frac{1}{2}}(\alpha_{ni}r)}{I_{n+\frac{1}{2}}(\alpha_{ni})}\right\}\right.\\&\left.+\frac{S_{ni}}{\sqrt{r}}\left\{\frac{J_{n-\frac{1}{2}}(\alpha_{ni}r)}{J_{n+\frac{1}{2}}(\alpha_{ni})}-\frac{I_{n-\frac{1}{2}}(\alpha_{ni}r)}{I_{n+\frac{1}{2}}(\alpha_{ni})}\right\}\right]P_{n-1}^m(\mu)\,\mathrm{e}^{\mathrm{i}(m\phi+pt)}\Bigg\},\end{aligned} \tag{4·5}$$

where
$$\left.\begin{aligned}K_{ni} &= -\frac{\lambda A_{ni}}{(n+1)\,(n+2)}\frac{\alpha_{ni}^2}{\alpha_{ni}^4-q_{n+1}^4}, & M_{ni} &= -\frac{\lambda A_{ni}}{(n+1)\,(n+2)}\frac{q_{n+1}^2}{\alpha_{ni}^4-q_{n+1}^4},\\ N_{ni} &= -\frac{\lambda A_{ni}}{n(n-1)}\frac{\alpha_{ni}^2}{\alpha_{ni}^4-q_{n-1}^4}, & S_{ni} &= -\frac{\lambda A_{ni}}{n(n-1)}\frac{q_{n-1}^2}{\alpha_{ni}^4-q_{n-1}^4}.\end{aligned}\right\} \tag{4·6}$$

It is clear that, since

$$\begin{aligned}T^{(p)}(1)-\xi T^{(p)\prime}(1) = \sum_{n=m}^{\infty}\sum_{i=1}^{\infty}\Bigg[&\frac{n(n+2)\,(n-m+1)}{2n+1}\{(1+\xi n+2\xi)\,\alpha_{ni}\beta_{ni}K_{ni}+\xi(n+1)\,\alpha_{ni}^2M_{ni}\}\\&\times P_{n+1}^m(\mu)-\frac{(n-1)\,(n+1)\,(n+m)}{2n+1}\{(1-\xi n+\xi)\,\alpha_{ni}\gamma_{ni}N_{ni}+\xi n\alpha_{ni}^2S_{ni}\}P_{n-1}^m(\mu)\Bigg]\mathrm{e}^{\mathrm{i}(m\phi+pt)},\end{aligned} \tag{4·7}$$

this solution does not in general satisfy (2·13). We must therefore add to $T^{(p)}$ a multiple, $T^{(c)}$, of the complementary function defined by equation (2·9), i.e.

$$\frac{1}{\sqrt{r}}\frac{J_{n+\frac{1}{2}}(q_nr)}{J_{n+\frac{1}{2}}(q_n)}P_n^m(\mu)\,\mathrm{e}^{\mathrm{i}(m\phi+pt)}.$$

Evidently, in order that

$$T(1)-\xi T'(1) \equiv [T^{(p)}(1)-\xi T^{(p)\prime}(1)]+[T^{(c)}-\xi T^{(c)\prime}(1)] = 0,$$

we must take

$$T^{(c)} = -\sum_{n=m}^{\infty}\sum_{i=1}^{\infty} \mathrm{e}^{\mathrm{i}(m\phi+pt)}$$

$$\times\Big[\frac{n(n+2)\,(n-m+1)}{2n+1}\Big\{\frac{(1+\xi n+2\xi)\,\alpha_{ni}\beta_{ni}K_{ni}+\xi(n+1)\,\alpha_{ni}^2 M_{ni}}{(1+\xi n+2\xi)\,J_{n+\frac{3}{2}}(q_{n+1})-\xi q_{n+1}J_{n+\frac{1}{2}}(q_{n+1})}\Big\}\frac{1}{\surd r}J_{n+\frac{3}{2}}(q_{n+1}r)\,P_{n+1}^m(\mu)$$

$$+\frac{(n-1)\,(n+1)\,(n+m)}{2n+1}\Big\{\frac{(1-\xi n+\xi)\,\alpha_{ni}\gamma_{ni}N_{ni}+\xi n\alpha_{ni}^2 S_{ni}}{(1-\xi n+\xi)\,J_{n-\frac{1}{2}}(q_{n-1})+\xi q_{n-1}J_{n+\frac{1}{2}}(q_{n-1})}\Big\}\frac{1}{\surd r}J_{n-\frac{1}{2}}(q_{n-1}r)\,P_{n-1}^m(\mu)\Big], \tag{4·8}$$

i.e.

$$T=\sum_{n=m}^{\infty}\sum_{i=1}^{\infty}\mathrm{e}^{\mathrm{i}(m\phi+pt)}\Big\{\frac{\alpha_{ni}n(n+2)\,(n-m+1)}{(2n+1)}$$

$$\times\Big[\frac{K_{ni}}{\surd r}\Big\{\frac{J_{n+\frac{3}{2}}(\alpha_{ni}r)}{J_{n+\frac{1}{2}}(\alpha_{ni})}-\frac{I_{n+\frac{3}{2}}(\alpha_{ni}r)}{I_{n+\frac{1}{2}}(\alpha_{ni})}-\frac{(1+\xi n+2\xi)\beta_{ni}J_{n+\frac{3}{2}}(q_{n+1}r)}{(1+\xi n+2\xi)\,J_{n+\frac{3}{2}}(q_{n+1})-\xi q_{n+1}J_{n+\frac{1}{2}}(q_{n+1})}\Big\}$$

$$+\frac{M_{ni}}{\surd r}\Big\{\frac{J_{n+\frac{3}{2}}(\alpha_{ni}r)}{J_{n+\frac{1}{2}}(\alpha_{ni})}+\frac{I_{n+\frac{3}{2}}(\alpha_{ni}r)}{I_{n+\frac{1}{2}}(\alpha_{ni})}-\frac{\xi(n+1)\,\alpha_{ni}J_{n+\frac{3}{2}}(q_{n+1}r)}{(1+\xi n+2\xi)\,J_{n+\frac{3}{2}}(q_{n+1})-\xi q_{n+1}J_{n+\frac{1}{2}}(q_{n+1})}\Big\}\Big]P_{n+1}^m(\mu)$$

$$-\frac{\alpha_{ni}(n-1)\,(n+1)\,(n+m)}{(2n+1)}$$

$$\times\Big[\frac{N_{ni}}{\surd r}\Big\{\frac{J_{n-\frac{1}{2}}(\alpha_{ni}r)}{J_{n+\frac{1}{2}}(\alpha_{ni})}+\frac{I_{n-\frac{1}{2}}(\alpha_{ni}r)}{I_{n+\frac{1}{2}}(\alpha_{ni})}-\frac{(1-\xi n+\xi)\,\gamma_{ni}J_{n-\frac{1}{2}}(q_{n-1}r)}{(1-\xi n+\xi)\,J_{n-\frac{1}{2}}(q_{n-1})+\xi q_{n-1}J_{n+\frac{1}{2}}(q_{n-1})}\Big\}$$

$$+\frac{S_{ni}}{\surd r}\Big\{\frac{J_{n-\frac{1}{2}}(\alpha_{ni}r)}{J_{n+\frac{1}{2}}(\alpha_{ni})}-\frac{I_{n-\frac{1}{2}}(\alpha_{ni}r)}{I_{n+\frac{1}{2}}(\alpha_{ni})}-\frac{\xi n\alpha_{ni}J_{n-\frac{1}{2}}(q_{n-1}r)}{(1-\xi n+\xi)\,J_{n-\frac{1}{2}}(q_{n-1})+\xi q_{n-1}J_{n+\frac{1}{2}}(q_{n-1})}\Big\}\Big]P_{n-1}^m(\mu)\Big\}. \tag{4·9}$$

5. The matrix form of the equation governing S

Our next step is to multiply equation (2·10) by $B_{ni}P_n^m\,\mathrm{e}^{-\mathrm{i}(m\phi+pt)}r^2\,\mathrm{d}r\,\mathrm{d}\mu\,\mathrm{d}\phi$, and integrate over the interior of the sphere. To this end, we first use our expressions (4·4) and (4·9) for Θ and T, to evaluate the contributions made by the right-hand side of equation (2·10).

On using (3·23) and (4·4), we see that the matrix element to which the term $L^2\Theta$ gives rise is

$$\iiint_{\text{sphere}} L^2\Theta[B_{ni}P_n^m\,\mathrm{e}^{-\mathrm{i}(m\phi+pt)}]\,r^2\,\mathrm{d}r\,\mathrm{d}\mu\,\mathrm{d}\phi$$

$$=\frac{4\pi(n+m)!}{(n-m)!}\frac{n^2(n+1)^2}{2n+1}\sum_{j=1}^{\infty}\frac{A_{nj}}{\alpha_{nj}^4-\chi^4}\int_0^1\Big[\chi^2B_{nj}+\alpha_{nj}^2C_{nj}-\frac{2\alpha_{nj}^2}{\surd r}\frac{J_{n+\frac{1}{2}}(\chi r)}{J_{n+\frac{1}{2}}(\chi)}\Big]B_{ni}r^2\,\mathrm{d}r. \tag{5·1}$$

By (3·14) to (3·16), we obtain

$$\iiint_{\text{sphere}} L^2\Theta[B_{ni}P_n^m\mathrm{e}^{-\mathrm{i}(m\phi+pt)}]\,r^2\,\mathrm{d}r\,\mathrm{d}\mu\,\mathrm{d}\phi$$

$$=\frac{4\pi(n+m)!}{(n-m)!}\frac{n^2(n+1)^2}{2n+1}\sum_{j=1}^{\infty}\frac{A_{nj}}{(\alpha_{nj}^4-\chi^4)}\Big[\delta_{ij}\chi^2a_{ni}+\alpha_{ni}^2\alpha_{nj}^2\Big(c_{nij}+\frac{2m_{ni}(\chi)}{m_n(\chi)}\Big)\Big]. \tag{5·2}$$

By (3·29) and (3·30), we see that, when the operator Q^3 is applied to the typical term of

(4·9), harmonics P^m_{n+2}, P^m_n and P^m_{n-2} result. We shall treat these terms separately, writing, in an obvious notation,

$$Q^3T = Q^3T_{(n+2)} + Q^3T_{(n)} + Q^3T_{(n-2)}. \tag{5.3}$$

By (3·27), (3·29) and (4·9), we obtain

$$\lambda Q^3 T_{(n+2)} = \sum_{n=m}^{\infty}\sum_{i=1}^{\infty} \frac{\lambda\alpha_{ni}^2 (n-m+1)(n-m+2)(n+3)!}{(2n+1)(2n+3)(n-1)!}$$
$$\times\Big[\frac{K_{ni}}{\sqrt{r}}\Big\{\frac{J_{n+\frac{5}{2}}(\alpha_{ni}r)}{J_{n+\frac{1}{2}}(\alpha_{ni})}+\frac{I_{n+\frac{5}{2}}(\alpha_{ni}r)}{I_{n+\frac{1}{2}}(\alpha_{ni})}-\frac{(1+\xi n+2\xi)\beta_{ni}q_{n+1}J_{n+\frac{5}{2}}(q_{n+1}r)}{\alpha_{ni}[(1+\xi n+2\xi)J_{n+\frac{3}{2}}(q_{n+1})-\xi q_{n+1}J_{n+\frac{1}{2}}(q_{n+1})]}\Big\}$$
$$+\frac{M_{ni}}{\sqrt{r}}\Big\{\frac{J_{n+\frac{5}{2}}(\alpha_{ni}r)}{J_{n+\frac{1}{2}}(\alpha_{ni})}-\frac{I_{n+\frac{5}{2}}(\alpha_{ni}r)}{I_{n+\frac{1}{2}}(\alpha_{ni})}-\frac{\xi(n+1)q_{n+1}J_{n+\frac{5}{2}}(q_{n+1}r)}{(1+\xi n+2\xi)J_{n+\frac{3}{2}}(q_{n+1})-\xi q_{n+1}J_{n+\frac{1}{2}}(q_{n+1})}\Big\}\Big]P^m_{n+2}\,\mathrm{e}^{\mathrm{i}(m\phi+pt)}. \tag{5·4}$$

On integration, using (3·16) to (3·18), we find

$$\iiint_{\text{sphere}} \lambda Q^3 T_{(n+2)}[B_{ni}P^m_n \mathrm{e}^{-\mathrm{i}(m\phi+pt)}]\,r^2\,\mathrm{d}r\,\mathrm{d}\mu\,\mathrm{d}\phi = \frac{4\pi\lambda(n+m)!\,(n+1)!}{(2n+1)(2n-1)(2n-3)(n-m-2)!\,(n-3)!}$$
$$\times\sum_{j=1}^{\infty}\Big\{K_{n-2,j}\Big[\frac{(2n-1)\alpha_{ni}^2\alpha_{n-2,j}}{\alpha_{ni}^4-\alpha_{n-2,j}^4}(\xi\alpha_{n-2,j}^3-\alpha_{ni}\gamma_{ni}\beta_{n-2,j})$$
$$-\frac{\alpha_{ni}^2\alpha_{n-2,j}\beta_{n-2,j}q_{n-1}(1+\xi n)}{\alpha_{ni}^4-q_{n-1}^4}\Big\{\frac{2q_{n-1}J_{n-\frac{1}{2}}(q_{n-1})-(\alpha_{ni}\gamma_{ni}-\xi q_{n-1}^2)J_{n+\frac{1}{2}}(q_{n-1})}{(1+\xi n)J_{n-\frac{1}{2}}(q_{n-1})-\xi q_{n-1}J_{n-\frac{3}{2}}(q_{n-1})}\Big\}\Big]$$
$$+M_{n-2,j}\Big[\frac{\alpha_{ni}^2\alpha_{n-2,j}^2}{(\alpha_{ni}^4-\alpha_{n-2,j}^4)}\{(2-2n\xi+\xi)\alpha_{ni}\gamma_{ni}+(2+2n\xi-\xi)\alpha_{n-2,j}\beta_{n-2,j}\}$$
$$-\frac{\alpha_{ni}^2\alpha_{n-2,j}^2q_{n-1}\xi(n-1)}{\alpha_{ni}^4-q_{n-1}^4}\Big\{\frac{2q_{n-1}J_{n-\frac{1}{2}}(q_{n-1})-(\alpha_{ni}\gamma_{ni}-\xi q_{n-1}^2)J_{n+\frac{1}{2}}(q_{n-1})}{(1+\xi n)J_{n-\frac{1}{2}}(q_{n-1})-\xi q_{n-1}J_{n-\frac{3}{2}}(q_{n-1})}\Big\}\Big]\Big\}. \tag{5·5}$$

On eliminating $K_{n-2,j}$ and $M_{n-2,j}$ in favour of $A_{n-2,j}$, by means of (4·6), we obtain, after some reductions,

$$\iiint_{\text{sphere}} \lambda Q^3 T_{(n+2)}[B_{ni}P^m_n \mathrm{e}^{-\mathrm{i}(m\phi+pt)}]\,r^2\,\mathrm{d}r\,\mathrm{d}\mu\,\mathrm{d}\phi$$
$$= -\frac{4\pi\mathbf{T}(n+m)!}{(n-m-2)!}\frac{(n-2)(n+1)}{(2n-3)(2n-1)(2n+1)}\sum_{j=1}^{\infty}\frac{\alpha_{ni}^2\alpha_{n-2,j}^2A_{n-2,j}}{\alpha_{ni}^4-\alpha_{n-2,j}^4}[(\alpha_{ni}\gamma_{ni}-\xi q_{n-1}^2)\,m_{n-2,j}(q_{n-1})$$
$$+(\alpha_{n-2,j}\beta_{n-2,j}+\xi q_{n-1}^2)\,p_{ni}(q_{n-1})+(2n-1)\,\xi-\xi(\alpha_{ni}^4-\alpha_{n-2,j}^4)f_{n-1}m_{n-2,j}(q_{n-1})\,p_{ni}(q_{n-1})]. \tag{5·6}$$

By an exactly similar calculation, we find that

$$\iiint_{\text{sphere}} \lambda Q^3 T_{(n-2)}[B_{ni}P^m_n \mathrm{e}^{-\mathrm{i}(m\phi+pt)}]\,r^2\,\mathrm{d}r\,\mathrm{d}\mu\,\mathrm{d}\phi$$
$$= -\frac{4\pi\mathbf{T}(n+m+2)!}{(n-m)!}\frac{n(n+3)}{(2n+1)(2n+3)(2n+5)}\sum_{j=1}^{\infty}\frac{\alpha_{n+2,j}^2\alpha_{ni}^2A_{n+2,j}}{\alpha_{n+2,j}^4-\alpha_{ni}^4}$$
$$\times[(\alpha_{n+2,j}\gamma_{n+2,j}-\xi q_{n+1})\,m_{ni}(q_{n+1})+(\alpha_{ni}\beta_{ni}+\xi q_{n+1}^2)\,p_{n+2,j}(q_{n+1})$$
$$+(2n+3)\,\xi-\xi(\alpha_{n+2,j}^4-\alpha_{ni}^4)f_{n+1}m_{ni}(q_{n+1})\,p_{n+2,j}(q_{n+1})]. \tag{5·7}$$

Similarly, we have

$$\iiint_{\text{sphere}} \lambda Q^3 T_{(n)} [B_{ni} P_n^m \,\mathrm{e}^{-\mathrm{i}(m\phi+pt)}]\, r^2\, \mathrm{d}r\, \mathrm{d}\mu\, \mathrm{d}\phi$$

$$= \frac{4\pi \mathbf{T}(n+m+1)!}{(n-m)!} \frac{n^2(n+2)(n-m+1)}{(n+1)(2n+1)^2(2n+3)} \sum_{j=1}^{\infty} \frac{\alpha_{ni}^2 \alpha_{nj}^2 A_{nj}}{\alpha_{nj}^4 - q_{n+1}^4} [\delta_{ij}\, a_{ni} + q_{n+1}^2 c_{nij}$$

$$+ (\alpha_{nj}\beta_{nj} + \xi q_{n+1}^2)\, m_{ni}(q_{n+1}) - \xi(\alpha_{nj}^4 - q_{n+1}^4) f_{n+1}\, m_{ni}(q_{n+1})\, m_{nj}(q_{n+1})]$$

$$+ \frac{4\pi \mathbf{T}(n+m)!}{(n-m-1)!} \frac{(n+1)^2(n-1)(n+m)}{n(2n+1)^2(2n-1)} \sum_{j=1}^{\infty} \frac{\alpha_{ni}^2 \alpha_{nj}^2 A_{nj}}{\alpha_{nj}^4 - q_{n-1}^4} [\delta_{ij}\, a_{ni} + q_{n-1}^2 c_{nij}$$

$$- (\alpha_{nj}\gamma_{nj} - \xi q_{n-1}^2)\, p_{ni}(q_{n-1}) - \xi(\alpha_{nj}^4 - q_{n-1}^4) f_{n-1}\, p_{ni}(q_{n-1})\, p_{nj}(q_{n-1})]. \tag{5·8}$$

It remains to evaluate the matrix elements of the left-hand side of (2·10). We find

$$\iiint_{\text{sphere}} [\nabla^2\{\nabla^2 L^2 S + i(m\lambda - pL^2)\, S\}]\, [B_{ni} P_n^m \,\mathrm{e}^{-\mathrm{i}(m\phi+pt)}]\, r^2\, \mathrm{d}r\, \mathrm{d}\mu\, \mathrm{d}\phi$$

$$= \frac{4\pi(n+m)!}{(n-m)!} \frac{n(n+1)}{2n+1} \sum_{j=1}^{\infty} A_{nj}\, \alpha_{nj}^2 \int_0^1 [\alpha_{nj}^2 B_{nj} - q_n^2 C_{nj}]\, B_{ni} r^2\, \mathrm{d}r$$

$$= \frac{4\pi(n+m)!}{(n-m)!} \frac{n(n+1)}{2n+1} \sum_{j=1}^{\infty} A_{nj}\, \alpha_{ni}^2 \alpha_{nj}^2 (\delta_{ij}\, a_{ni} - q_n^2 c_{nij}). \tag{5·9}$$

Equation (2·10) may now be replaced by the infinite set of linear equations

$$\sum_{l=m}^{\infty} \sum_{j=1}^{\infty} \Pi_{ni:lj} A_{lj} = 0, \tag{5·10}$$

where i runs from 1 to ∞, and n from m to ∞. The matrix elements, Π, are zero except for the cases listed in equations (5·11) to (5·13) below. It may be noticed that two completely distinct families of solutions exist: one in which $l-m$ and $n-m$ are non-negative even integers (>0, if $m=0$), and one for which they are positive odd integers. For $m=0$, the former is the more fundamental in the sense that it appears to be associated with the lower critical Rayleigh number, for given m and $\mathbf{T}$. For $m \neq 0$, the reverse is true. We shall call the solutions the even and the odd modes, respectively. We have

$$\frac{\Pi_{ni:ni}}{\alpha_{ni}^4} = a_{ni} - q_n^2 c_{nii} - \frac{Rn(n+1)}{\alpha_{ni}^4 - \chi^4} \left[\frac{\chi^2 a_{ni}}{\alpha_{ni}^4} + c_{nii} + \frac{2m_{ni}(\chi)}{m_n(\chi)}\right] + \frac{n(n+2)(n+m+1)(n-m+1)\mathbf{T}}{(n+1)^2(2n+1)(2n+3)(\alpha_{ni}^4 - q_{n+1}^4)}$$

$$\times [a_{ni} + q_{n+1}^2 c_{nij} + (\alpha_{ni}\beta_{ni} + \xi q_{n+1}^2)\, m_{ni}(q_{n+1}) - \xi(\alpha_{ni}^4 - q_{n+1}^4) f_{n+1}\, m_{ni}^2(q_{n+1})]$$

$$+ \frac{(n+1)(n-1)(n+m)(n-m)\mathbf{T}}{n^2(2n+1)(2n-1)(\alpha_{ni}^4 - q_{n-1}^4)} [a_{ni} + q_{n-1}^2 c_{nii} - (\alpha_{ni}\gamma_{ni} - \xi q_{n-1}^2)\, p_{ni}(q_{n-1})$$

$$- \xi(\alpha_{ni}^4 - q_{n-1}^4) f_{n-1}\, p_{ni}^2(q_{n-1})]. \tag{5·11}$$

If $i \neq j$, we have

$$\frac{\Pi_{ni:nj}}{\alpha_{ni}^2 \alpha_{nj}^2} = -q_n^2 c_{nij} - \frac{Rn(n+1)}{\alpha_{nj}^4 - \chi^4} \left[c_{nij} + \frac{2m_{ni}(\chi)}{m_n(\chi)}\right] + \frac{n(n+2)(n+m+1)(n-m+1)\mathbf{T}}{(n+1)^2(2n+1)(2n+3)(\alpha_{nj}^4 - q_{n+1}^4)}$$

$$\times [q_{n+1}^2 c_{nij} + (\alpha_{nj}\beta_{nj} + \xi q_{n+1}^2)\, m_{ni}(q_{n+1}) - \xi(\alpha_{nj}^4 - q_{n+1}^4) f_{n+1}\, m_{ni}(q_{n+1})\, m_{nj}(q_{n+1})]$$

$$+ \frac{(n+1)(n-1)(n+m)(n-m)\mathbf{T}}{n^2(2n+1)(2n-1)(\alpha_{nj}^4 - q_{n-1}^4)} [q_{n-1}^2 c_{nij} - (\alpha_{nj}\gamma_{nj} - \xi q_{n-1}^2)\, p_{ni}(q_{n-1})$$

$$- \xi(\alpha_{nj}^4 - q_{n-1}^4) f_{n-1}\, p_{ni}(q_{n-1})\, p_{nj}(q_{n-1})], \tag{5·12}$$

and

$$\frac{(n+1)(2n-1)}{(n-1)(n-m)(n-m+1)} \Pi_{n+1,i:n-1,j} = \frac{n(2n+3)}{(n+2)(n+m)(n+m+1)} \Pi_{n-1,j:n+1,i}$$

$$= -\frac{\alpha_{n+1,i}^2 \alpha_{n-1,j}^2 \mathbf{T}}{(2n+1)(\alpha_{n+1,i}^4 - \alpha_{n-1,j}^4)} [(\alpha_{n+1,i}\gamma_{n+1,i} - \xi q_n^2) m_{n-1,j}(q_n) + (\alpha_{n-1,j}\beta_{n-1,j} + \xi q_n^2) p_{n+1,i}(q_n)$$

$$+ (2n+1)\xi - \xi(\alpha_{n+1,i}^4 - \alpha_{n-1,j}^4) f_n p_{n+1,i}(q_n) m_{n-1,j}(q_n)]. \quad (5\cdot13)$$

6. Numerical results: small T

The solution of the infinite set of equations (5·10) has been investigated by truncating the series expansion (3·1) after h values of i and q values of n (starting at $n = 1$ or 2 if $m = 0$ and at $n = m$ or $m+1$ otherwise, and rising by 2). The result is qh linear homogeneous equations in qh unknowns which have a non-trivial solution for $R = R_{qh}$, say. The hope and belief, well supported by the numerical work, is that, as $q \to \infty$ and $h \to \infty$, R_{qh} tends to the required eigenvalue, R, of the infinite set of equations. Since the different harmonics n are separable when $\mathbf{T} = 0$, it is natural that, when $\mathbf{T}$ is small, best numerical results are obtained, for (approximately) given qh, by taking q small and h large. The axisymmetric steady case ($m = p = 0$) bring this out clearly. For, in this case as Chandrasekhar (1957) has shown, the formulation has a variational basis, and $R_{qh} \to R$ from above as q and $h \to \infty$. Thus a smaller value of R_{qh} is necessarily a better approximation to R than a larger value. It was found for example that, for the free odd modes at $\mathbf{T} = 10^3$, $R_{6,4} = 8981\cdot15$ while $R_{3,8} = 8979\cdot44$. For large $\mathbf{T}$, on the other hand, it seems best to choose q and h to be roughly of the same size. It was found for example that, for the free odd modes at $\mathbf{T} = 4\times10^5$, $R_{8,9} = 168\,908$, while $R_{6,13} = 169\,568$ and $R_{10,7} = 168\,954$. The experience acquired was later used in choosing the best values of q and h in the axisymmetric case as a guide in the non-axisymmetric case in which a similar variational principle does not exist.

The steady axisymmetric case could also be used to provide information about the rate of convergence of R_{qh} to R which is useful in the non-axisymmetric and overstable ($p \neq 0$) cases. In these, the matrix elements (5·11) to (5·13) are complex, so that, after truncating at the level (q, h), equations (5·10) provide $2qh$ real equations. Thus, with given machine storage,† it is only possible to proceed 'half as far'. Assuming, however, that the convergence of R_{qh} in these cases is roughly the same as that of the steady axisymmetric case, we can use the latter to estimate the error in the complex cases. It was found for example that, although convergence was rapid for small $\mathbf{T}$, the values of R_{qh} for $qh \approx 40$ were approximately 3% greater than those obtained for $qh \approx 80$ when $\mathbf{T} = 10^6$, but were approximately 10% greater when $\mathbf{T} = 4\times10^6$.

The results for R in the steady axisymmetric case are given, as functions of $\mathbf{T}$ in tables 1 and 2, and in figure 1. It may be seen that the numerical work agrees well with our asymptotic results (paper I, see also § 7 below) for the case $\mathbf{T} \to \infty$. Even the magnitude and negative sign of the first ($\mathbf{T}^{-\frac{1}{12}}$) correction term for the rigid case were well substantiated. The agreement with the result of Bisshopp & Niiler (1965, equation 6·11) is, however, less

† The KDF 9 computer of the University of Newcastle limited the matrices to 80×80 real elements.

satisfactory. It is also interesting to note that despite the accuracy† of the matrix elements given by Bisshopp (1958), his estimates of R appear to be significantly too large for $\mathbf{T} > 10^4$. This may be attributed to a too severe truncation in his representation

TABLE 1. AXISYMMETRIC STEADY CONVECTION (RIGID BOUNDARY)

$\mathbf{T}$	R (odd mode)	R (even mode)
10^3	$9{\cdot}32952 \times 10^3$	$1{\cdot}2442 \times 10^4$
4×10^3	$1{\cdot}22825 \times 10^4$	$1{\cdot}7994 \times 10^4$
10^4	$1{\cdot}64495 \times 10^4$	$2{\cdot}6366 \times 10^4$
4×10^4	$2{\cdot}98385 \times 10^4$	$4{\cdot}9054 \times 10^4$
10^5	$4{\cdot}74689 \times 10^4$	$7{\cdot}6903 \times 10^4$
4×10^5	$1{\cdot}02414 \times 10^5$	$1{\cdot}6212 \times 10^5$
10^6	$1{\cdot}7760 \times 10^5$	$2{\cdot}758 \times 10^5$
4×10^6	$4{\cdot}3534 \times 10^5$	$6{\cdot}775 \times 10^5$
10^7	$8{\cdot}38 \times 10^5$	

TABLE 2. AXISYMMETRIC STEADY CONVECTION (FREE BOUNDARY)

$\mathbf{T}$	R (odd mode)	R (even mode)
10^3	$8{\cdot}97940 \times 10^3$	$8{\cdot}9541 \times 10^3$
4×10^3	$1{\cdot}67367 \times 10^4$	$1{\cdot}8244 \times 10^4$
10^4	$2{\cdot}53151 \times 10^4$	$2{\cdot}8391 \times 10^4$
4×10^4	$4{\cdot}90395 \times 10^4$	$5{\cdot}7601 \times 10^4$
10^5	$7{\cdot}89398 \times 10^4$	$9{\cdot}4326 \times 10^4$
4×10^5	$1{\cdot}68907 \times 10^5$	$2{\cdot}0572 \times 10^5$
10^6	$2{\cdot}8727 \times 10^5$	$3{\cdot}522 \times 10^5$
4×10^6	$6{\cdot}6937 \times 10^5$	$8{\cdot}955 \times 10^5$

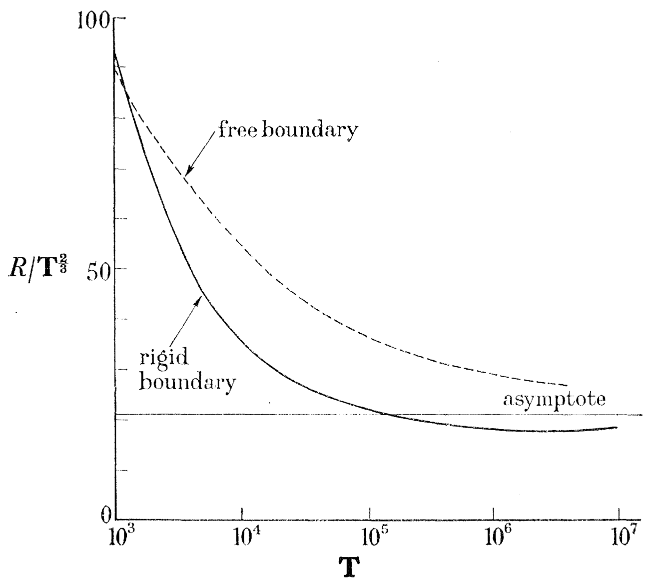

FIGURE 1. Axisymmetric steady convection. $R/\mathbf{T}^{\frac{2}{3}}$ is shown as a function of $\mathbf{T}$ for the cases of rigid and free boundaries. The asymptote $R/\mathbf{T}^{\frac{2}{3}} = 20{\cdot}7126$ for $\mathbf{T} \to \infty$ is shown.

† The entries given in tables 5 to 7 by Bisshopp (1958) appear to be correct to within 3 in the final place shown, with the following exceptions: in the rigid case, the $\mathbf{R}$ entry $n = m = 4$, $j = k = 2$ should be $0{\cdot}48627$ and the $\mathbf{T}$ entry for $n = m = 4$, $j = k = 3$ should be $1{\cdot}27028$; in the free case, the $\mathbf{R}$ entry for $n = m = 0$, $j = k = 2$ should be $-0{\cdot}48175$ and for $n = m = 2$, $j = k = 3$ should be $-8{\cdot}6841$. Apart from a transpositional error, the roots shown in his table 3 are correct to within 1 in the last place with one exception which, however, is correct to within 3 in the last place. We may note here that Bisshopp's analytic work appears to contain two very minor errors which, however, could not have persisted in his numerical work: (i) in his equation (85) for $\langle nk|FT|nj\rangle$, the first term in curly brackets should be multiplied by 2, and (ii) in his equation (87) for $\langle mj|FQ|nk\rangle$, the over-all sign should be reversed.

series. It is, nevertheless, remarkable that he could obtain such accurate results by desk calculator.

In the non-axisymmetric and overstable cases, p is a second eigenvalue to be determined in addition to R, by the demand that both the real and imaginary parts of the determinant of Π coefficients vanish. The computation of these matrix elements required the evaluation of spherical Bessel functions of semi-imaginary argument. For modulus of argument less than five times the order, these were evaluated by series; for larger moduli the asymptotic expansions were used. Nevertheless, for large n (~ 16), there was some loss of accuracy ($\sim 10^{-5}$ %) if order and modulus of argument were roughly equal. The whole programme could be partially tested by repeating the axisymmetric cases, and the agreement was satisfactory. As we explained above, we could not expect results derived from this programme to exceed 10 % accuracy at $\mathbf{T} = 4 \cdot 10^6$. We did not, in fact, consider values of $\mathbf{T}$ larger than 10^5.

Table 3. Asymmetric modes (rigid boundary)

$\mathbf{T}$	$R\ (m=1)$	$R\ (m=2)$	$R\ (m=3)$	$R\ (m=4)$	$R\ (m=5)$
10^3	9680	11070	15507	22234	31443
3×10^3	12346	12316	16271	22770	31848
10^4	18081	15788	18625	24498	33189
3×10^4	27557	22353	23781	28617	36555
10^5	45531	35135	35423	39023	45701

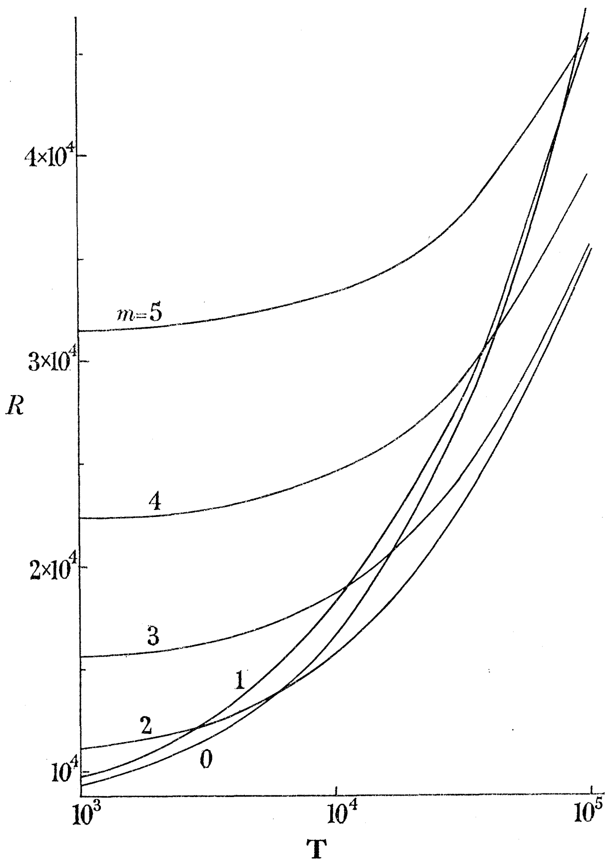

Figure 2. Asymmetric convection. R is shown as a function of $\mathbf{T}$ for the case of rigid boundaries.

Our results for R in the non-symmetric cases (rigid boundary only) are given in table 3 and in figure 2. It will be seen that the value of m for which R is least increases systematically with $\mathbf{T}$. This result is supported by the asymptotic analysis of §§7 to 9.

7. The nearly axisymmetric case in the limit $\mathbf{T} \to \infty$

In this section we begin a study of the limit $\mathbf{T} \to \infty$, extending further the analysis of paper I. It will be appropriate to suppose that, as the limit $\mathbf{T} \to \infty$ is taken,

$$m = M\mathbf{T}^{\alpha}. \tag{7·1}$$

We first concentrate on fixed values for the constants M and α, and try to determine which such mode is 'preferred', i.e. which is associated with the smallest value, R_c, of R. Later, we will minimize R_c itself over M and α, thereby obtaining the 'overall' preferred mode.

We may conveniently divide the possible choice of α into five subclasses (i) $\alpha < \frac{1}{6}$, (ii) $\alpha = \frac{1}{6}$, (iii) $\frac{1}{6} < \alpha < \frac{1}{4}$, (iv) $\alpha = \frac{1}{4}$, and (v) $\alpha > \frac{1}{4}$. We discuss case (i) in this section, case (ii) in § 8, and the remainder in § 9.

We suppose, following paper I, that

$$\frac{\partial}{\partial \rho} = O(\mathbf{T}^{\frac{1}{6}}), \quad \frac{\partial}{\partial z} = O(1), \quad \frac{\partial}{\partial t} = p = O(m\mathbf{T}^{\frac{1}{6}}), \tag{7·2}$$

where (ρ, ϕ, z) are cylindrical polar coordinates in which Oz is $\theta = 0$, the axis of rotation. As explained in paper I, the first of (7·2) is not really an assumption; rather it defines a 'range of interest' in which the minimum, R_c, is believed to lie. This belief can be tested at the conclusion of the analysis and, if incorrect, a new postulate chosen. The last of (7·2) is an assumption which can be verified *a posteriori*. It is clearly incorrect in overstable cases ($m = 0$, $p \neq 0$) in which it should be replaced by $p = O(\mathbf{T}^{\frac{1}{3}})$. These have, however, been studied in paper I and will be examined no further here.

From (7·2) we have, to leading order,

$$\left.\begin{aligned} \nabla^2 &\approx \frac{\partial}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2}, \quad L^2 \approx m^2 - z^2\nabla^2, \\ Q^3 &\approx -m^2\frac{\partial}{\partial z} + z\left(z\frac{\partial}{\partial z} + 1\right)\nabla^2. \end{aligned}\right\} \tag{7·3}$$

The action of these modified operators on $J_m(\zeta\rho)$ is clearly algebraic, i.e. solutions proportional to $J_m(\zeta\rho)$ exist and, as explained in paper I, these in fact provide a basis for the required asymptotic solutions. (Here ζ is a constant.) For solutions proportional to $J_m(\zeta\rho)$, we have, by (7·3)

$$\left.\begin{aligned} \nabla^2 &\approx -\zeta^2, \quad L^2 \approx m^2 + \zeta^2 z^2, \\ Q^3 &\approx -\left[(m^2+\zeta^2 z^2)\frac{\partial}{\partial z} + \zeta^2 z\right], \end{aligned}\right\} \tag{7·4}$$

and (2·9) to (2·11) become, respectively,

$$[(m^2+\zeta^2 z^2)(\zeta^2+\mathrm{i}p) - \mathrm{i}m\lambda]\, T = \lambda\left[(m^2+\zeta^2 z^2)\frac{\partial}{\partial z} + \zeta^2 z\right] S, \tag{7·5}$$

$$\zeta^2[(m^2+\zeta^2 z^2)(\zeta^2+\mathrm{i}p) - \mathrm{i}m\lambda]\, S = \lambda\left[(m^2+\zeta^2 z^2)\frac{\partial}{\partial z} + \zeta^2 z\right] T + R(m^2+\zeta^2 z^2)\,\Theta, \tag{7·6}$$

$$(\zeta^2 + \mathrm{i}\omega p)\,\Theta = (m^2+\zeta^2 z^2)\, S. \tag{7·7}$$

These equations may be solved in the form

$$\Theta = \frac{m^2+\zeta^2z^2}{\zeta^2+\mathrm{i}\omega p}S = \zeta^2(\zeta^2+\mathrm{i}p)\,zF+\mathrm{i}m\lambda F', \tag{7·8}$$

$$T = \frac{\zeta^2+\mathrm{i}\omega p}{m^2+\zeta^2z^2}\left[\lambda\zeta^2zF'+\mathrm{i}m\left\{\zeta^2(\zeta^2+\mathrm{i}p)-\frac{R(m^2+\zeta^2z^2)}{\zeta^2+\mathrm{i}\omega p}\right\}F\right], \tag{7·9}$$

where $F(z)$ satisfies

$$\mathrm{d}^2F/\mathrm{d}z^2 = (A-Bz^2)\,F, \tag{7·10}$$

and

$$A = \frac{\zeta^2}{\mathbf{T}}(\zeta^2+\mathrm{i}p)^2+\frac{\mathrm{i}mR}{\mathbf{T}^{\frac{1}{2}}(\zeta^2+\mathrm{i}\omega p)}-\frac{m^2R(\zeta^2+\mathrm{i}p)}{\mathbf{T}(\zeta^2+\mathrm{i}\omega p)}, \tag{7·11}$$

$$B = \frac{\zeta^2R(\zeta^2+\mathrm{i}p)}{\mathbf{T}(\zeta^2+\mathrm{i}\omega p)}. \tag{7·12}$$

All these results were derived, in different fashion, in paper I, where it was also shown that F is proportional to u_z.

Naturally, since the system of equations (7·8) to (7·12) governing the 'main stream' part of the asymptotic solution is only of second order, while the original problem was of eighth order, we cannot subject F to conditions (2·13) in their entirety. As is usual in this type of singular perturbation problem, the correct way of obtaining the leading approximation to the main stream is to subject F to the boundary condition of lowest differential order, leaving to suitably constructed boundary layers the task of satisfying the remainder (see paper I, § III). Thus we require that $S = 0$ on $r = 1$, i.e. by (7·8)

$$\zeta^2(\zeta^2+\mathrm{i}p)\,zF+\mathrm{i}m\mathbf{T}^{\frac{1}{2}}F' = 0, \quad \text{on} \quad r = 1. \tag{7·13}$$

Of course, (7·13) cannot be obeyed simultaneously everywhere on $r = 1$. This, however, as will presently become apparent, is not in fact required.

If $0 < \alpha < \frac{1}{6}$, the solution can, as we will presently see, be regarded as a slight perturbation of the steady axisymmetric state. According to (7·10) to (7·13), this is governed by

$$\mathrm{d}^2F_0/\mathrm{d}z^2 = (A_0-B_0z^2)\,F_0, \tag{7·14}$$

where

$$A_0 = \zeta^6/\mathbf{T}, \quad B_0 = \zeta^2R_0/\mathbf{T}, \tag{7·15}$$

and

$$F_0(\pm 1) = 0. \tag{7·16}$$

The justification for applying (7·16) at the poles is given in paper I [see also argument beneath (7·22) below]. Briefly, it rests on the fact that, by the first of (7·2), $\zeta = O(\mathbf{T}^{\frac{1}{6}})$ in the range of interest. Thus the required solution, which is proportional to $J_0(\zeta\rho)$, vanishes (compared with its values on the axis) outside an axial cell of radius $\mathbf{T}^{-\frac{1}{6}}$. Naturally (7·13) should be applied at the ends of this cell.

In addition to (7·16), we apply

$$F_0'(0) = 0, \quad F_0(0) = 1, \tag{7·17}$$

the second of which is purely a normalization condition. The first selects the 'odd' mode mentioned below (5·10), which has a smaller Rayleigh number than the 'even' mode.

As both Roberts (paper I) and Bisshopp & Niiler (1965) have shown, (7·14) to (7·17) are equivalent to the variational problem $\delta R_0 = 0$, where

$$B_0 \int_0^1 z^2 F_0^2 \,\mathrm{d}z = \int_0^1 \left(\frac{\mathrm{d}F_0}{\mathrm{d}z}\right)^2 \mathrm{d}z + A_0 \int_0^1 F_0^2 \,\mathrm{d}z. \tag{7·18}$$

If we suppose a trial function of the form

$$F_0 = \sum_{n=1}^{N} a_n \cos\left(n-\tfrac{1}{2}\right)\pi z, \tag{7·19}$$

and minimize over a_n, we obtain a set of N simultaneous equations for a_1 to a_N, which have a non-trivial solution if† $\det C_{mn} = 0$, where

$$C_{mn} = \delta_{mn}\left[\frac{\zeta^2 R_0}{\mathrm{T}}\left(\frac{1}{3} - \frac{2}{(2n+1)^2\pi^2}\right) - \frac{\zeta^6}{\mathrm{T}} - \left(n+\tfrac{1}{2}\right)^2\pi^2\right] + (1-\delta_{mn})\left[\frac{2\zeta^2 R_0}{\mathrm{T}}(-1)^{m-n}\frac{(2m+1)(2n+1)}{(m+n+1)^2(m-n)^2\pi^2}\right]. \tag{7·20}$$

TABLE 4. EIGENVALUE (VARIATIONAL METHOD)

N	$\zeta_c/\mathrm{T}^{\frac{1}{6}}$	$R_c/\mathrm{T}^{\frac{2}{3}}$
1	1·035623	26·404757
2	1·099711	20·965063
3	1·107586	20·729482
4	1·108187	20·714565
5	1·108250	20·712970
6	1·108260	20·712699
7	1·108262	20·712636
∞	1·108263	20·712604

The values of R_{0c} and ζ_{0c} (the value of ζ at which R_{0c} occurs), are shown in table 4 as a function of the degree of truncation, N. It is clear that the convergence is rapid. The final ($N = \infty$) entry of the table gives the result of the exact numerical integration reported in paper I.‡

Returning now to the case $0 < \alpha \leqslant \frac{1}{6}$, we first observe that

$$\frac{J_m(\zeta\rho)}{J_m(m)} = \frac{J_m(m\rho/\rho_c)}{J_m(m)}, \tag{7·21}$$

where
$$\rho_c = m/\zeta. \tag{7·22}$$

By the well-known asymptotic properties of Bessel functions, the right of (7·21) vanishes algebraically as $m \to \infty$ for fixed $\rho/\rho_c > 1$, and exponentially for fixed $\rho/\rho_c < 1$, the 'distance' in which the function falls to zero in each case being $O(m^{-\frac{2}{3}})$. Thus, for $\rho_c \ll m^{-\frac{2}{3}}$, i.e. for $\alpha < \frac{1}{10}$, the solutions obtained above, vanish outside an axial cell of radius $O(\mathrm{T}^{\alpha-\frac{1}{6}})$. For $\alpha > \frac{1}{10}$, they vanish everywhere except in a cylindrical annulus coaxial with the rotation

† The corresponding result by Bisshopp & Niiler (1965, eqn. 6·5) appears to be in error by a factor of 2 in the off-diagonal elements of C. Comparing their findings with those of table 4 above, it appears that this mistake persisted in their numerical work.

‡ This opportunity is taken of correcting three misprints in paper I. Above (57), z_c should read 1·3612213, while on that line (and also on p. 250), $F'(z_c)$ should be replaced by $f'(z_c)$.

axis, of radius ρ_c and of thickness $O(\mathbf{T}^{-2\alpha/3})$. In this case, we may replace (7·13) by

$$\zeta^2(\zeta^2+\mathrm{i}p)\,zF+\mathrm{i}m\mathbf{T}^{\frac{1}{3}}F' = 0, \quad \text{on} \quad z = (1-m^2/\zeta^2)^{\frac{1}{2}}, \tag{7·23}$$

i.e. by the condition that u_r vanishes on the intersection of the annular cell with the surface of the sphere.

Until this moment, the analysis has been valid not only in case (i) but also in case (ii). Now, however, we postpone further discussion of case (ii) until § 8. We suppose $\alpha < \frac{1}{6}$, and present the problem, posed by (7·23) and (7·10) to (7·12), as an expansion about the solution of (7·14) to (7·16) in a series of powers of $\mathrm{i}m\mathbf{T}^{-\frac{1}{6}}$; we write

$$A = A_0+\left(\frac{\mathrm{i}m}{\mathbf{T}^{\frac{1}{6}}}\right)A_1+\left(\frac{\mathrm{i}m}{\mathbf{T}^{\frac{1}{6}}}\right)^2 A_2, \quad \text{etc.} \tag{7·24}$$

We readily discover, for the marginal case, that

$$p = \mathbf{T}^{\frac{1}{3}}\left[\frac{sm}{\mathbf{T}^{\frac{1}{6}}}+O\left(\frac{m^3}{\mathbf{T}^{\frac{1}{2}}}\right)\right], \quad R = \mathbf{T}^{\frac{2}{3}}\left[\lambda_0-\Lambda\left(\frac{m^2}{\mathbf{T}^{\frac{1}{3}}}\right)+O\left(\frac{m^4}{\mathbf{T}^{\frac{2}{3}}}\right)\right], \tag{7·25}$$

where $\lambda_0 = R_0\,\mathbf{T}^{-\frac{2}{3}}$, and s and Λ are eigenvalues to be determined. According to (7·11) and (7·12), we have

$$\left.\begin{aligned} A_1 &= \frac{2s\zeta^4}{\mathbf{T}^{\frac{2}{3}}}+\frac{R_0}{\zeta^2\mathbf{T}^{\frac{1}{3}}}, & A_2 &= \frac{\zeta^2 s^2}{\mathbf{T}^{\frac{1}{3}}}-\frac{\omega s R_0}{\zeta^4}+\frac{R_0}{\mathbf{T}^{\frac{2}{3}}}, \\ B_1 &= (1-\omega)\frac{sR_0}{\mathbf{T}^{\frac{2}{3}}}, & B_2 &= \frac{\zeta^2\Lambda}{\mathbf{T}^{\frac{1}{3}}}-\frac{\omega(1-\omega)\,s^2R_0}{\zeta^2\mathbf{T}^{\frac{1}{3}}}, \end{aligned}\right\} \tag{7·26}$$

and, by equating like powers of $\mathrm{i}m\,\mathbf{T}^{-\frac{1}{6}}$ in (7·10), we obtain (7·14) and

$$\frac{\mathrm{d}^2F_1}{\mathrm{d}z^2} = (A_0-z^2B_0)\,F_1+(A_1-z^2B_1)\,F_0, \tag{7·27}$$

$$\frac{\mathrm{d}^2F_2}{\mathrm{d}z^2} = (A_0-z^2B_0)\,F_2+(A_1-z^2B_1)\,F_1+(A_2-z^2B_2)\,F_0. \tag{7·28}$$

In a similar way, (7·23) may be expanded to give rise to (7·16) and to

$$F_1(1) = -\frac{\mathbf{T}^{\frac{2}{3}}}{\zeta^4}F_0'(1), \quad F_2(1) = -\frac{\mathbf{T}^{\frac{2}{3}}}{\zeta^4}F_1'(1)+\left(\frac{s\mathbf{T}}{\zeta^6}-\frac{\mathbf{T}^{\frac{1}{3}}}{2\zeta^2}\right)F_0'(1). \tag{7·29}$$

The required F is even; we may also choose $F = 1$ at $z = 0$. Then F, like F_0, obeys (7·17), and their difference obeys

$$F_1(0) = F_1'(0) = F_2(0) = F_2'(0) = 0. \tag{7·30}$$

Let G_0 be a second independent solution of (7·14) for the same values of A_0 and B_0 as F_0; for definiteness, suppose it obeys

$$G_0'(0) = 1, \quad G_0(0) = 0. \tag{7·31}$$

Clearly, by (7·14) and the first of (7·17) and (7·31), the Wronskian of F_0 and G_0 is

$$F_0\,G_0'-G_0\,F_0' = 1, \tag{7·32}$$

which, incidentally, shows that $$G_0 = F_0\int_0^z \frac{\mathrm{d}z}{F^2}.$$

By (7·16) and (7·32) we have

$$\frac{1}{G_0(1)} = -F_0'(1) = 4{\cdot}02528. \tag{7·33}$$

Here we have used the solution for the case $\zeta = 1{\cdot}10826\mathrm{T}^{\frac{1}{6}}$ for which λ_0 is least. In this connexion it should be observed that the variational property (7·18) assures us that this value of ζ will also provide the smallest value of R in the nearly axisymmetric case considered here.

We may write down the formal solution of (7·27) and the first pair of (7·30) as

$$F_1 = -F_0\int_0^z (A_1-z^2B_1)\,F_0\,G_0\,\mathrm{d}z + G_0\int_0^z (A_1-z^2B_1)\,F_0^2\,\mathrm{d}z. \tag{7·34}$$

This satisfies the first of (7·29) if

$$\int_0^1 (A_1-z^2B_1)\,F_0^2\,\mathrm{d}z = \frac{\mathrm{T}^{\frac{2}{3}}}{\zeta^4}[F_0'(1)]^2. \tag{7·35}$$

This, with the aid of (7·26), determines s as a function of ω, the integrals

$$\int_0^1 F_0^2\,\mathrm{d}z = 0{\cdot}87445, \quad \int_0^1 z^2F_0^2\,\mathrm{d}z = 0{\cdot}19107, \tag{7·36}$$

being easily calculable properties of the known solution F_0.

On multiplying (7·28) by F_0 and integrating over the range (0, 1) we find, after an integration by parts, that

$$\int_0^1 (A_2-z^2B_2)\,F_0^2\mathrm{d}z + \int_0^1 (A_1-z^2B_1)\,F_0\,F_1\,\mathrm{d}z = -F_0'(1)\,F_2(1). \tag{7·37}$$

Using (7·34) and (7·29) this may be written

$$\int_0^1 (A_2-z^2B_2)\,F_0^2\,\mathrm{d}z + 2\int_0^1 \mathrm{d}z(A_1-z^2B_1)\,F_0\,G_0\int_0^z (A_1-z^2B_1)\,F_0^2\,\mathrm{d}z$$
$$= -\left(\frac{\mathrm{T}^{\frac{1}{6}}}{\zeta}\right)^8 [F_0'(1)]^2\left[F_0'(1)\,G_0'(1) - \frac{s\zeta^2}{\mathrm{T}^{\frac{1}{3}}} + \frac{\zeta^6}{2\mathrm{T}}\right]. \tag{7·38}$$

Now $G_0'(1) = -2{\cdot}30887$, and the integrals

$$\left.\begin{aligned}\int_0^1 \mathrm{d}z\,F_0\,G_0\int_0^z F_0^2\,\mathrm{d}z = 0{\cdot}19628, \quad \int_0^1 \mathrm{d}z\,F_0\,G_0\int_0^z z^2F_0\,G_0\,\mathrm{d}z = 0{\cdot}02598,\\ \int_0^1 z^2\,\mathrm{d}z\,F_0\,G_0\int_0^z F_0^2\,\mathrm{d}z = 0{\cdot}081007, \quad \int_0^1 z^2\,\mathrm{d}zF_0\,G_0\int_0^z z^2F_0\,G_0\,\mathrm{d}z = 0{\cdot}013010,\end{aligned}\right\} \tag{7·39}$$

are easily calculable properties of F_0 and G_0. Thus, by (7·36) and (7·39), Λ may be determined as a function of ω. The results are given† in table 5. It will be noted that, in all cases Λ is positive. This strongly indicates (cf. equation (7·25)) that the overall preferred mode is more asymmetric than those covered by this theory, i.e. the smallest value of R is in the range $\alpha \geqslant \frac{1}{6}$. This, as we will presently see, is indeed so.

Before leaving the present case, we should observe that the Λ term of (7·25) does not

† It may be noted that s is infinite at $\omega = \frac{1}{3}$, at which the present perturbation approach is, presumably, invalid. The reason for this is not clear.

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necessarily provide the dominant correction to R_0. Two other effects may be greater. First, the fact that the width of the annular cell is $O(m^{-\frac{2}{3}})$ means that (7·13) is not exactly satisfied across its intersection with the sphere. When this is allowed for, a correction $O(m^{-\frac{4}{3}}\mathbf{T}^{\frac{2}{3}})$ is added to R_0. Secondly, boundary layers are required to deal with the final conditions of

Table 5. Values of s and Λ

ω	s	Λ
0·025	−3·2828	5·899
0·1	−4·3381	33·079
0·3	−30·3668	4940·030
1	0·5183	51·203
3	0·3796	15·394
10	0·1047	9·086
40	0·0255	7·437

(2·13), which have not yet been satisfied. This, as was shown in section III of paper I, introduces a term of order $\mathbf{T}^{\frac{7}{12}}$ into the expansion of R, if the boundary is rigid, and of order $\mathbf{T}^{\frac{1}{3}}$, if it is free. In the leading approximation, this correction is unmodified by the slight degree of asymmetry considered here, i.e. it is independent of m.

8. The limit $\mathbf{T} \to \infty$ for intermediate asymmetries

The important case $\alpha = \frac{1}{6}$ has already been formulated in (7·10) to (7·12) and (7·23), above. Convection occurs in an annular cell, of thickness $\mathbf{T}^{-\frac{1}{6}}$, whose radius $\rho_c = m/\zeta = O(1)$. The process of solutions is entirely straightforward: For each fixed value of M (cf. equation (7·1)), we seek the value ζ_M (say) of ζ for which R is least ($= R_M$, say). As in case (i), $\zeta_M = O(\mathbf{T}^{\frac{1}{6}})$ and $R_M = O(\mathbf{T}^{\frac{2}{3}})$, while $p_M = O(\mathbf{T}^{\frac{1}{3}})$. We find that, as M increases from 0 to ∞, ρ_c increases from 0 to 1, i.e. the cell, from being on the axis in the axisymmetric case, gradually becomes equatorial. Also, $R_M/\mathbf{T}^{\frac{2}{3}}$ decreases systematically until a minimum is reached, and then increases systematically. For $\omega = 1$, for example, the minimum value 14·1304254 of

Table 6. Critical asymmetries

ω	$m/\mathbf{T}^{\frac{1}{6}}$	θ_c°	$R_c/\mathbf{T}^{\frac{2}{3}}$	$p/\mathbf{T}^{\frac{1}{3}}$
0·025	0·202	34·5	0·2018	7·21
0·1	0·310	34·1	1·22	4·29
0·3	0·412	33·3	4·64	2·53
1	0·531	32·3	14·13	0·877
3	0·664	33·1	18·31	0·151
10	0·716	34·2	18·88	0·0345
40	0·729	34·4	19·01	0·0080

$R_M/\mathbf{T}^{\frac{2}{3}}$ occurs when $M = 0·53144$ and when $\theta_c = 32·310°$. [Here $\theta_c = \sin^{-1}\rho_c$ denotes the colatitude of the intersection of the cell with the surface.] The behaviour of $R_M/\mathbf{T}^{\frac{2}{3}}$ as a function of M for the case $\omega = 1$ is shown in figure 3. Results for the minima are also given in table 6. The case of small ω deserves special comment. Here, as we have shown in paper I, overstable oscillations are possible and, indeed, the critical value of $R/\mathbf{T}^{\frac{2}{3}}$ for them vanishes with ω, when $\omega \to 0$. The numerical results for the non-symmetric case suggest that their critical $R/\mathbf{T}^{\frac{2}{3}}$ also vanishes with ω, when $\omega \to 0$; and indeed that they are always smaller than the corresponding overstable Rayleigh numbers. This indicates that, even when overstable solutions are possible, they are not preferred. It appears likely that, at all Prandtl numbers the smallest Rayleigh number belongs to an unsymmetric mode.

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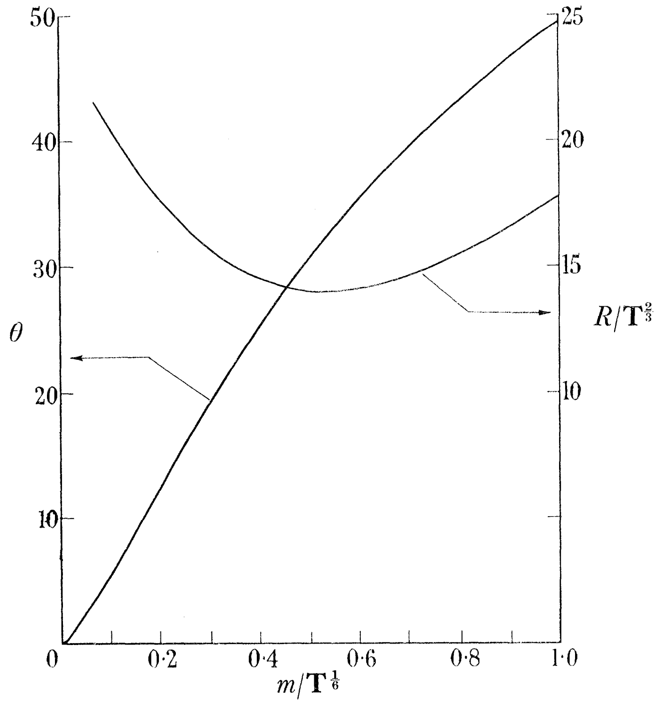

FIGURE 3. Asymptotic results for $\mathbf{T} \to \infty$ in the case in which $m = O(\mathbf{T}^{\frac{1}{6}})$, and $\omega = 1$. Convection occurs in a cylindrical shell, coaxial with $\boldsymbol{\Omega}$, intersecting the surface of the sphere at a co-latitude of θ (given here in degrees). It will be seen that $R/\mathbf{T}^{\frac{2}{3}}$ has a minimum, for which the radius of the convection cell is about half the radius of the sphere.

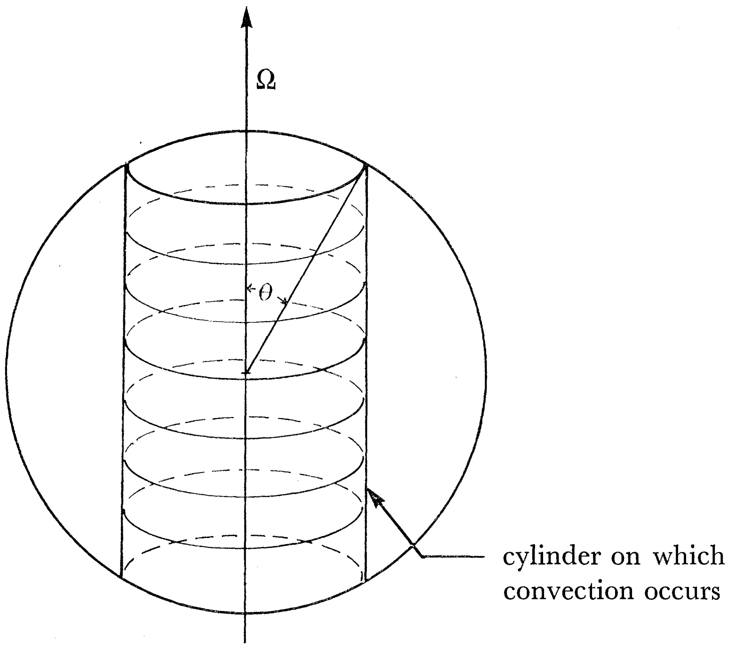

FIGURE 4

9. THE HIGHLY ASYMMETRIC CASE IN THE LIMIT $\mathbf{T} \to \infty$

For completeness, we now discuss, rather briefly, the remaining possible values of α. The case $\alpha > \frac{1}{4}$ is particularly simple. Here rotation plays no part in the leading term of the asymptotic solution which, therefore, becomes the solution in the non-rotating case (cf.,

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for example, Roberts 1965*b*),

$$S = S(r)\, P_l^m(\mu)\, e^{im\phi}, \tag{9·1}$$

where S satisfies

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right]^3 S = -l(l+1)\, RS, \tag{9·2}$$

$$S = (1-\xi)\, S' + \xi S'' = \left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right]^2 S = 0, \quad \text{on} \quad r = 1. \tag{9·3}$$

The smallest value (m) of l yields the smallest value of R for given m. We therefore replace l by m in (9·1) to (9·3) and, since $m \gg 1$, replace $m(m+1)$ by m^2.

In the limit $R \to \infty$, (9·2) shows that convection occurs over the entire surface of the sphere, but that it penetrates only a small distance within it. To see this, write

$$r = 1-\epsilon\zeta, \tag{9·4}$$

where $\epsilon \to 0$ as $m \to \infty$. Then, to leading order, (9·2) becomes

$$\frac{3m^4}{\epsilon^2}\frac{d^2S}{d\zeta^2} - m^6\,[1+6\epsilon\zeta]\, S = -m^2RS,$$

i.e.

$$\frac{d^2S}{d\zeta^2} = 2\epsilon^3 m^2(\zeta-\zeta_s)\, S, \tag{9·5}$$

where

$$\zeta_s = \frac{1}{6\epsilon m^4}(R-m^4). \tag{9·6}$$

According to (9·5), convection penetrates the sphere to a depth of order $m^{-\frac{2}{3}}$; in fact, a convenient choice of ϵ is seen to be

$$\epsilon = (2m^2)^{-\frac{1}{3}}. \tag{9·7}$$

Then (9·5) gives

$$S \propto \mathrm{Ai}(\zeta-\zeta_s); \tag{9·8}$$

where Ai denotes the Airy function of the first kind. The other independent solution to (9·5) involves the Airy function of the second kind and is exponentially large as $\zeta \to \infty$; this is physically unacceptable. Since the equation (9·5) governing the main stream is only of second order, we require (in the usual way) that it obeys only the first of (9·3), i.e. we take

$$\mathrm{Ai}\,(-\zeta_s) = 0. \tag{9·9}$$

The smallest root, 2·33810741, of this equation yields the smallest value of R, namely

$$R = m^4 + 11{\cdot}1345 m^{10/3}. \tag{9·10}$$

In the case $\frac{1}{6} < \alpha < \frac{1}{4}$, we have $p = O(m\mathbf{T}^{\frac{1}{3}})$, as before. Thus again the time derivatives in (2·9) to (2·11) are, in the first approximation, negligible in comparison with the ∇^2 terms with which they appear and which are $O(m^2)$. Moreover, the difference between $L^2(\partial/\partial z)$ and $(\partial/\partial z)\, L^2$ is negligible in the first approximation, so that (2·12) may be rewritten

$$Q^3 \approx -L^2\frac{\partial}{\partial z}. \tag{9·11}$$

Elimination of T and Θ in (2·9) to (2·11) now gives

$$\nabla^6 S + \mathbf{T}\frac{\partial^2 S}{\partial z^2} + RL^2 S = 0. \tag{9·12}$$

The remarkable similarity between this equation and that governing convection in a plane rotating layer (cf. Chandrasekhar 1961, chap. III, equation (99)) should be noted; it provides some *a posteriori* justification for the qualitative discussion of § 1.

The solution of (9·12) to be described is in the form of an equatorial cell, i.e. convection only occurs near the equator $r = 1, \theta = \frac{1}{2}\pi$ of the sphere and penetrates only a small distance in the θ direction and a small distance from the surface. For this reason it is advantageous to introduce a change of variable, replacing ρ by

$$\rho^* = 1 - \frac{\rho}{\sqrt{(1-z^2)}}. \tag{9·13}$$

The surface of the sphere then becomes $\rho^* = 0$. In the first approximation,

$$\left.\begin{aligned} \nabla^2 &\approx -m^2\left[1+z^2+2\rho^* - \frac{1}{m^2}\frac{\partial^2}{\partial \rho^{*2}}\right], \\ L^2 &\approx m^2(1+z^2), \end{aligned}\right\} \tag{9·14}$$

so that (9·12) gives, to leading order,

$$\left\{-m^6\left(1+3z^2+6\rho^* - \frac{3}{m^2}\frac{\partial^2}{\partial\rho^{*2}}\right) + \mathbf{T}\frac{\partial^2}{\partial z^2} + Rm^2(1+z^2)\right\} S = 0. \tag{9·15}$$

This equation admits separable solutions:

$$S = Y(\rho^*)\, Z(z), \tag{9·16}$$

for, on substituting (9·16) into (9·15), we obtain

$$\left\{\frac{\mathbf{T}}{m^6}\frac{Z''}{Z} - \left(3 - \frac{R}{m^4}\right) z^2\right\} + 3\left\{\frac{Y''}{m^2 Y} - 2\rho^*\right\} + \left(\frac{R}{m^4} - 1\right) = 0. \tag{9·17}$$

(We shall see presently that $R = m^4$ in the leading approximation. We may therefore replace the R/m^4 in the first bracket by 1. This procedure would, however, be incorrect in the final term where the z^2 of the first term does not appear and the difference between R/m^4 and 1, though small, is of the same order as the remaining terms of (9·17), as we will soon see.)

Let us introduce boundary-layer variables ζ and τ to replace ρ^* and z:

$$\rho^* = \frac{\zeta}{(2m^2)^{\frac{1}{3}}}, \qquad z = \left(\frac{\mathbf{T}}{8m^6}\right)^{\frac{1}{4}} \tau. \tag{9·18}$$

Then (9·17) becomes

$$\frac{(8\mathbf{T})^{\frac{1}{2}}}{m^3}\left[\frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}\tau^2} - \tfrac{1}{4}\tau^2\right] + \frac{6}{(2m^2)^{\frac{1}{3}}}\left[\frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}\zeta^2} - \zeta\right] + \left[\frac{R}{m^4} - 1\right] = 0. \tag{9·19}$$

If we denote the separation constant by $\gamma + \frac{1}{2}$, we have

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\tau^2} + (\gamma + \tfrac{1}{2} - \tfrac{1}{4}\tau^2)\, Z = 0. \tag{9·20}$$

This is Weber's equation which has solutions bounded both as $\tau \to +\infty$ and $\tau \to -\infty$ if, and only if, γ is a non-negative integer, n (say). The solutions are then proportional to the well-known Hermite functions:

$$D_n(\tau) = (-1)^n \mathrm{e}^{\frac{1}{4}\tau^2} \frac{\mathrm{d}^n}{\mathrm{d}\tau^n}(\mathrm{e}^{-\frac{1}{2}\tau^2}). \tag{9·21}$$

Now (9·19) becomes

$$\frac{d^2Y}{d\zeta^2} = (\zeta - \zeta_s)\, Y, \tag{9·22}$$

where [in place of definition (9·6)], we have

$$\zeta_s = \frac{2(R-m^4)}{3(2m^2)^{\frac{4}{3}}} - \frac{2(2n+1)}{3(2m^2)^{\frac{7}{6}}}\,\mathrm{T}^{\frac{1}{2}}. \tag{9·23}$$

Following the argument given in the case $\alpha > \frac{1}{4}$, we see that (9·8) and (9·9) hold, and that the smallest value of R is given by $n = 0$ and by the smallest zero of (9·9). We obtain

$$R = m^6 + 11{\cdot}1345 m^{\frac{10}{3}} + 1{\cdot}4142 m \mathrm{T}^{\frac{1}{2}}. \tag{9·24}$$

It is interesting to observe that, according to (9·24), $m \to 0$ provide the smallest value of R. This indicates, in support of §§ 7 and 8, that the overall preferred mode is less asymmetric than those considered in this section, i.e. the smallest value of R lies in the range $\alpha \leqslant \frac{1}{6}$. This diminishes our interest in the final (difficult) case (iv), $\alpha = \frac{1}{4}$. We have not examined this case in detail, but it appears that as M increases from zero to ∞, the equatorial cell of case (iii) gradually spreads over the surface of the sphere to become the surface convection of case (v).

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